ETBS204-MODULE – V

COMPLEX INTEGRATION

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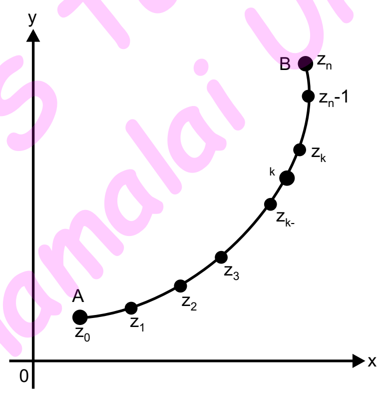
5 Complex Integration

5.1 Introduction

Let $w = f(z)$ be a continuous function of the complex variable z and C be any continuous curve connecting the two points A and B of the domain. Divide C into n parts by the points $z_1, z_2, z_3 \cdots z_{n-1}$ where A & B correspond to z_0 & z_n.

Let $\Delta z_k = z_k - z_{k-1}$ and let α_k be any arbitrary point in the arc Δz_k.

Then, consider the sum $S_n = \sum\limits_{k=1}^{n} f(\alpha_k)\Delta z_k$



If the limits S_n exists as $n \to \infty$ and each $z_k \to 0$ and if this limit is independent of the mode of subdivision of C and the choice of α_k, then it is defined as the line integral of $f(z)$ from A to B along C. it is denoted by

$$\int_C f(z)dz \qquad \text{or} \qquad \int_{AB} f(z)dz$$

$$\text{if} \quad w = f(z) = u(x,y) + iv(x,y)$$

$$\text{then} \quad \int_C f(z)dz = \int_C (u+iv)(dx+idy)$$

$$= \int_C (udx - vdy) + i \int_C (vdx + udy)$$

■ **Note:**

1. If the parametric equation of C be $x = \phi_1(t), y = \phi_2(t)$ then $w = f(z)$ & $dz = dx + idy$ can be expressed in terms of the parameter t.

   $$\text{i.e.,} \quad \int_C f(z)dz = \int_{t=a}^{b} f(z(t))\frac{dz}{dt}dt$$

2. $|\Delta z_k|$ denotes the length of the chord joining Z_{k-1} & Z_k. If the sum $\sum\limits_{k=1}^{n} |\Delta z_k|$ tends to a limit as $n \to \infty$ in such a way that each $\Delta z_k \to 0$, then the limit is called the length of the curve C.

3. If the starting point A of the arc and the end point B of the arc coincide then the complex line integral is called *closed contour integral* & is denoted by $\oint\limits_C f(z)dz$.

It does not indicate the direction along the curve but by convention we take anti-clock wise direction to be positive unless stated otherwise.

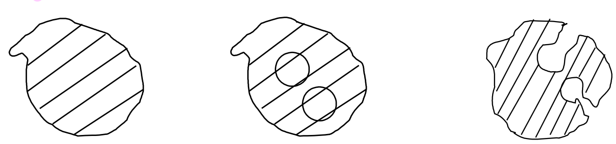
Regions

A region is a connected set of points. It may consist of interior points or interior and boundary points. An *open region* consists only the interior points, but a *closed region* consists all its boundary points and interior points.

A region R is said to be *connected* region if any two points of it are connected by curves, lying entirely with in the region.

A region R is said to be *simply connected* if any closed curve which lies in R can be shrunk to a point without passing out of the region.

A region which is not simply connected is called *multiply - connected*.



Simply connected region   Multiply connected region   Multiply connected region converted to a simple connected region

A multiply connected region can be converted to a simply connected region by introducing one or more cross cuts as shown in the above figure.

Properties

1. If $C = C_1 + C_2$ then $\int\limits_C f(z)dz = \int\limits_{C_1} f(z)dz + \int\limits_{C_2} f(z)dz$
2. If α be a complex constant, then $\int\limits_C \alpha f(z)dz = \alpha \int\limits_C f(z)dz$
3. $\int\limits_C [\alpha f(z) + g(z)]dz = \alpha \int\limits_C f(z)dz + \int\limits_C g(z)dz$
4. If C_2 represents the contour C_1, transferred in the reverse direction, then $\int\limits_{C_2} f(z)dz = -\int\limits_{C_1} f(z)dz$
5. If ℓ be the length of the curve C and if $|f(z)| \leq M$ for all z on C, then $\left|\int\limits_C f(z)dz\right| \leq M\ell$.

5.2 Cauchy's Integral Theorem (Cauchy's Fundamental Theorem)

Statement : If $f(z)$ is analytic and $f'(z)$ is continuous at every point inside and on a simple closed curve C then

$$\int\limits_C f(z)dz = 0$$

Proof:

Let $f(z) = u(x, y) + iv(x, y)$

$$\begin{aligned}\int\limits_C f(z)dz &= \int\limits_C (u + iv)(dx + idy) \\ &= \int\limits_C (udx - vdy) + i\int\limits_C udy + vdx\end{aligned}$$

Since $f'(z)$ is continuous, the four partial derivatives $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}$ and $\dfrac{\partial v}{\partial y}$ are also continuous on C and in the region R enclosed by C, then applying Green's theorem in a plane, which states that

$$\int\limits_C Pdx + Qdy = \iint\limits_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy$$

Since $f(z)$ is analytic, the Cauchy - Riemann equations hold at every point of R.

i.e., $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$, and $\dfrac{\partial u}{\partial y} = \dfrac{-\partial v}{\partial x}$

$$\int_C f(z)dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\ dy + i \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\ dy$$
$$= \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dx\ dy + i \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right) dx\ dy$$
$$= 0 + i0 = 0.$$

■ **Note :**

1. In the above form of Cauchy theorem we had the assumption that the function $f'(z)$ is continuous. It was the famous mathematician Goursat who first established that the above condition of continuity of $f'(z)$ is unnecessary and can be removed from the hypothesis.

   Hence if $f(z)$ is analytic at all points on and inside a simple closed curve C, then $\int_C f(z)dz = 0$.

2. **Extension of Cauchy's integral theorem**

   If $f(z)$ is analytic within and on a multiply connected region bounded by a simple closed curve C and non-intersecting close curves $C_1, C_2, \cdots C_n$ then

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \cdots + \int_{C_n} f(z)dz$$

   where all the integrals are taken in the same sense.

5.3 Cauchy's Integral Formula (Cauchy's Fundamental Formula)

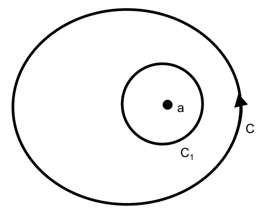
Statement: If $f(z)$ is an analytic function within and on a closed curve C of a simply connected region R and if 'a' is any point within C then

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$$

Proof :

Since $f(z)$ is analytic within and on C, therefore $\dfrac{f(z)}{z-a}$ is also analytic within and on C, except at the point $z = a$. Now draw a small circle C_1, with centre at $z = a$ and radius ρ, lying completely inside C.

The function $\frac{f(z)}{z-a}$ is analytic in the multiply connected region bounded by C and C_1. Applying Cauchy's extended theorem, we have



$$\int\limits_C \frac{f(z)}{z-a}dz = \int\limits_{C_1} \frac{f(z)}{z-a}dz$$

on $\quad C_1, |z-a| = \rho \quad$ hence $\quad z - a = \rho e^{i\theta}$

$$\text{or} \quad z = \rho e^{i\theta} + a$$

$$\therefore \quad dz = i\rho e^{i\theta} d\theta$$

Hence $\displaystyle \int\limits_C \frac{f(z)}{z-a}dz = \int\limits_{C_1} \frac{f(z)}{z-a}dz = \int\limits_0^{2\pi} \frac{f(a+\rho e^{i\theta})}{\rho e^{i\theta}} i\rho e^{i\theta} d\theta$

$$= \int\limits_0^{2\pi} f(a+\rho e^{i\theta}) d\theta \qquad \text{taking limit as } \rho \to 0$$

$$= i\int\limits_0^{2\pi} f(a) d\theta$$

$$= i\, f(a)[\theta]_0^{2\pi} = 2\pi i f(a)$$

$$\therefore \quad f(a) = \frac{1}{2\pi i}\int\limits_C \frac{f(z)}{z-a}dz$$

Cauchy's Integral Formula for Derivatives of an Analytic Function

(AU 2009)

By Cauchy's integral formula, we have

$$f(a) = \frac{1}{2\pi i}\int\limits_C \frac{f(z)}{z-a}dz$$

Differentiating partially both sides with respect to a within the integral sign, we get

$$f'(a) = \frac{1!}{2\pi i}\int\limits_C \frac{f(z)}{(z-a)^2}dz$$

Proceeding in a similar manner, we get

$$f''(a) = \frac{2!}{2\pi i}\int\limits_C \frac{f(z)}{(z-a)^3}dz$$

and in general

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz$$

■ **Note :**

Cauchy's integral formula can be extended to multiply connected region.

If $f(z)$ is analytic in the region bounded by closed contour $C_1, C_2, C_3, C_4 \cdots$ and 'a' is point in the region then

$$f(a) = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{(z-a)} dz - \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{(z-a)} dz - \frac{1}{2\pi i} \int_{C_3} \frac{f(z)}{(z-a)} dz \cdots$$

Example 5.1

Evaluate $\int\limits_0^{1+i} \left(x^2 - iy\right) dz$ along the path $y = x$. **(AU 2009)**

Solution:

Along $y = x$ $\quad dy = dx \quad$ x varies from 0 to 1

$$\int_0^{1+i} \left(x^2 - iy\right) dz = \int_0^{1+i} \left(x^2 - iy\right)(dx + idy)$$

$$= \int_0^1 \left(x^2 - ix\right)(dx - idx)$$

$$= \int_0^1 \left(x^2 - ix\right)(1 - 1i)\, dx$$

$$= (1+i)\left[\frac{x^3}{3} - i\frac{x^2}{2}\right]_0^1$$

$$= (1+i)\left(\frac{1}{3} - \frac{i}{2}\right)$$

$$= \frac{5}{6} - \frac{1}{6}i$$

> **Note:**
> $z = x + iy$
> $dz = dx + idy$
> put $y = x$
> & $dy = dx$

Example 5.2

Evaluate $\int_C (x^2 - iy^2) dz$ along

(i) the parabola $y = 2x^2$ from $(1, 2)$ to $(2, 8)$

(ii) the straight line from $(1, 1)$ to $(2, 8)$

Solution:

(i) Given $y = 2x^2$

$\therefore dy = 4x\,dx$

$$\therefore \quad \int_C (x^2 - iy^2)dz = \int_C (x^2 - iy^2)(dx + idy)$$

$$= \int_C (x^2 - i4x^4)(dx + i4xdx)$$

$$= \int_{x=1}^{2} (x^2 - i4x^4)(1 + i4x)dx$$

$$= \int_{x=1}^{2} (x^2 + 16x^5) + i(4x^3 - 4x^4)dx$$

$$= \left[\left(\frac{x^3}{3} + \frac{16x^6}{6}\right) + i\left(\frac{4x^4}{4} - \frac{4x^5}{5}\right)\right]_1^2 = \frac{511}{3} - \frac{49}{5}i$$

(ii) the straight line from $(1,1)$ to $(2,8)$ in parametric form is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} = t$$

$$\therefore \quad \frac{y-1}{8-1} = \frac{x-1}{2-1} = t \qquad \text{or} \qquad y - 1 = 7t$$

$$x - 1 = t$$

$$\therefore \qquad dy = 7dt \quad \text{and} \quad dx = dt \quad \text{and} \quad 0 \le t \le 1$$

$$\int_C (x^2 - iy^2)dz = \int_C \left[(t+1)^2 + i(7t+1)^2\right][dt + i7dt]$$

$$= \int_C \left[(t+1)^2 + i(7t+1)^2\right][1 + 7i]\,dt$$

$$= \int_{t=0}^{1} \left[(1+t)^2 - 7(7t+1)^2\right] + i\left[(7t+1)^2 + i7(t+1)^2\right]dt$$

$$= \left[\left[\frac{(1+t)^3}{3} - \frac{(7t+1)^3}{3}\right] + i\left[\frac{1}{7}\frac{(7t+1)^3}{3} + \frac{7(t+1)^3}{3}\right]\right]_0^1$$

$$= \frac{518}{3} - 8i$$

Example 5.3

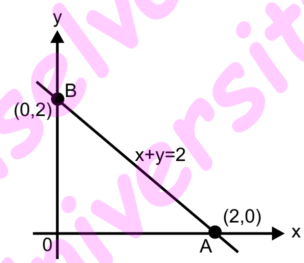
Evaluate $\int\limits_C (z^2 + 3z)dz$ along

(i) the straight line from $(2, 0)$ to $(0, 2)$

(ii) the straight lines $(2, 0)$ to $(2, 2)$ and then from $(2, 2)$ to $(0, 2)$.

Solution:

The equation to the straight line joining $(2, 0)$ to $(0, 2)$
is $x + y = 2$ or $y = 2 - x$ and $dy = -dx$



$$\therefore \int\limits_C (z^2 + 3z)dz$$

$$= \int\limits_C [(x + iy)^2 + 3(x + iy)](dx + idy)$$

$$= \int\limits_C (x^2 - y^2 + 2ixy + 3x + 3iy)(dx + idy)$$

$$= \int\limits_C [x^2 - (2 - x)^2 + 2ix(2 - x) + 3x + 3i(2 - x)](1 - i)dx$$

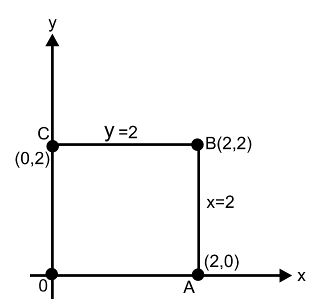
$$= \int_2^0 [(-2x^2 + 8x + 2) + i(-2x^2 - 6x + 10)]dx$$

$$= \left[\left(\frac{-2x^3}{3} + \frac{8x^2}{2} + 2x\right) + i\left(\frac{-2x^3}{3} - \frac{6x^2}{2} + 10x\right)\right]_2^0$$

$$= -\left[\frac{44}{3} + \frac{8}{3}i\right]$$

(ii) $$\int\limits_C (z^2 + 3z)dz = \int\limits_{\substack{AB \\ x=2 \\ dx=0}} (z^2 + 3z)dz$$

$$+ \int\limits_{\substack{BC \\ y=2 \\ dy=0}} (z^2 + 3z)dz$$



$$= \int_{y=0}^{2} [(2+iy)^2 + 3(2+iy)]idy + \int_{x=2}^{0} [(x+i2)^2 + 3(x+i2)]dx$$

$$= \left[\frac{(2+iy)^3}{3} + \frac{3\,(2+iy)^2}{2}\right]_0^2 + \left[\frac{(x+2i)^3}{3} + \frac{3\,(x+i2)^2}{2}\right]_0^2$$

$$= \left(\frac{-8}{3} - 6\right) - \frac{8}{3}i - 6$$

$$= -\frac{44}{3} - \frac{8}{3}i$$

Example 5.4

Evaluate $\int_0^{1+i} (x^2 + iy)dz$ along the parabola

(i) $y = x^2$ and (ii) $x = y^2$

Solution:

(i) Given $y = x^2$ $\qquad z = x + iy$

$\therefore \qquad dy = 2x\,dx \qquad \therefore$ when $z = 0 \;;\; x = 0$

$z = 1 + i \;;\; x = 1$

$$\therefore \int_0^{1+i} (x^2 + iy)dz = \int_0^1 (x^2 + ix^2)(dx + i2xdx)$$

$$= \int_0^1 x^2(1+i)(1+i2x)dx$$

$$= \int_0^1 \left[x^2(1-2x) + ix^2(1+2x)\right] dx$$

$$= \int_0^1 \left[(x^2 - 2x^3) + i(x^2 + 2x^3)\right] dx$$

$$= \left[\left(\frac{x^3}{3} - \frac{2x^4}{4}\right) + i\left(\frac{x^3}{3} + \frac{2x^4}{4}\right)\right]_0^1$$

$$= \left[\frac{1}{3} - \frac{1}{2}\right] + i\left[\frac{1}{3} + \frac{1}{2}\right]$$

$$= -\frac{1}{6} + i\frac{5}{6}$$

(ii) Given $x = y^2$ $\qquad z = x + iy$

$\therefore \quad dx = 2y\,dy \qquad \therefore$ when $z = 0 \;;\; x = 0$

$z = 1 + i \;;\; x = 1$

$$\therefore \int_0^{1+i} (x^2 + iy)dz = \int_0^1 (y^4 + iy)(2ydy + idy)$$

$$= \int_0^1 (y^4 + iy)(2y + i)dy$$

$$= \int_0^1 \left[(2y^5 - y) + i(y^4 + 2y^2)\right] dy$$

$$= \left[\left(\frac{2y^6}{6} - \frac{y^2}{2}\right) + i\left(\frac{y^5}{5} + \frac{2y^3}{3}\right)\right]_0^1$$

$$= \left[\frac{1}{3} - \frac{1}{2}\right] + i\left[\frac{1}{5} + \frac{2}{3}\right]$$

$$= -\frac{1}{6} + i\frac{13}{15}$$

Example 5.5

Evaluate $\int_C \frac{dz}{z-a}$ where C is the circle $|z - a| = r$.

Solution:

Given C is $|z - a| = r$

$$\therefore \quad z = a + re^{i\theta} \qquad \text{and} \qquad dz = re^{i\theta} i\, d\theta$$

$$\therefore \quad \int_C \frac{dz}{z-a} = \int_0^{2\pi} \frac{re^{i\theta} i\, d\theta}{a + re^{i\theta} - a} = i[\theta]_0^{2\pi} = 2\pi i$$

Example 5.6

Evaluate $\int_c \sec z\, dz$, where c is the unit circle $|z| = 1$. **(AU 2009)**

Solution:

$$\int_c \sec z \, dz = \int_c \frac{1}{\cos z} \, dz$$

$$\cos z = 0$$

$$\Rightarrow \quad z = (2n+1)\frac{\pi}{2}, \quad n = 0, 1, 2 \ldots$$

Singular points are $z = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots.$

Since the singular points are lies outside the circle $|z| = 1$.

$$\int_c \frac{dz}{\cos z} = \int_c \sec z \, dz = 0$$

Example 5.7

Evaluate $\int_0 (z-a)^m dz, m \neq -1, C$ is the circle $|z-a| = r$.

Solution:

Given C is $|z-a| = r$

$$\therefore \quad z = a + re^{i\theta}, \quad dz = re^{i\theta} i \, d\theta$$

$$\therefore \quad \int_C (z-a)^m dz = \int_0^{2\pi} r^m e^{im\theta} re^{i\theta} i d\theta$$

$$= r^{m+1} i \left[\frac{e^{(m+1)i\theta}}{(m+1)i} \right]_0^{2\pi}$$

$$= \frac{r^{m+1}}{m+1} \left[e^{2\pi i(m+1)} - 1 \right] = 0 \quad \text{Since } \; m \neq -1$$

Example 5.8

Evaluate $\int_C \frac{z+2}{z} dz$, where C is the semicircle $z = 2e^{i\theta}, 0 \leq \theta \leq \pi$.

Solution:

Given $z = 2e^{i\theta}$ $\quad \therefore\ dz = 2ie^{i\theta}d\theta$

$$\therefore \int_C \frac{z+2}{z}dz = \int_0^{\pi} \frac{(2e^{i\theta}+2)}{2e^{i\theta}}(2e^{i\theta}\ i\ d\theta)$$

$$= 2i\int_0^{\pi}(1+e^{i\theta})d\theta$$

$$= 2i\left[\left(\theta + \frac{e^{i\theta}}{i}\right)\right]_0^{\pi}$$

$$= 2i\left[\frac{\pi i - 2}{i}\right]$$

$$= 2\pi i - 4$$

Example 5.9

Evaluate $\int_C (z^2+3z)dz$ along the circle $|z| = 2$ from $(2,0)$ to $(0,2)$.

Solution:

Given $|z| = 2$, i.e. $z = 2e^{i\theta}$ and $0 \le \theta \le \frac{\pi}{2}$

$$\therefore \int_C (z^2+3z)dz = \int_{|z|=2} (z^2+3z)dz$$

$$= \int_0^{\pi/2}\left[(2e^{i\theta})^2 + 3(2e^{i\theta})\right]2e^{i\theta}.id\theta$$

$$= \left[8i\frac{e^{3i\theta}}{3i} + 12i\frac{e^{2i\theta}}{2i}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{8}{3}(e^{i3\pi/2}-1) + 6(e^{i\pi}-1)$$

$$= \frac{8}{3}(-i-1) + 6(-2)$$

$$= \frac{-44}{3} - \frac{8}{3}i$$

Example 5.10

Evaluate $\int_0^{2+i} z^2 dz$ along

(i) the line $x = 2y$

(ii) the real axis upto to 2 and then to $2 + i$.

Solution:

(i) Given $x = 2y$

$\therefore \quad dx = 2dy$

$$\begin{aligned}
\therefore \quad \int_0^{2+i} z^2 dz &= \int_0^{2+i} (x+iy)^2 (dx + idy) \\
&= \int_0^{2+i} (x^2 - y^2 + i2xy)(dx + idy) \\
&= \int_0^1 \left[4y^2 - y^2 + i2(2y)y\right] [2dy + idy] \\
&= \int_0^1 \left[3y^2 + i4y^2\right] [2 + i]\, dy \\
&= \int_0^1 \left(6y^2 - 4y^2\right) + i\left(3y^2 + 8y^2\right) dy \\
&= \int_0^1 \left[2y^2 + i11y^2\right] dy \\
&= \left[2\left(\frac{y^3}{3}\right) + i11\left(\frac{y^3}{3}\right)\right]_0^1 \\
&= \frac{2}{3} + i\frac{11}{3}
\end{aligned}$$

(ii) Now $\int_0^{2+i} z^2 dz = \int_{OA} z^2 dz + \int_{AB} z^2 dz$

$$= \int\limits_{\substack{OA \\ y=0 \\ dy=0}} z^2 dz + \int\limits_{\substack{AB \\ x=2 \\ dx=0}} z^2 dz$$

$$= \int\limits_{\substack{OA \\ y=0 \\ \therefore dy=0}} (x+iy)^2(dx+idy) + \int\limits_{\substack{AB \\ x=2 \\ \therefore dx=0}} (x+iy)^2(dx+idy)$$

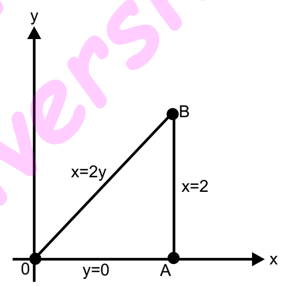
$$= \int\limits_{x=0}^{2} x^2 dx + \int\limits_{y=0}^{1} (2+iy)^2 i dy$$

$$= \left[\frac{x^3}{3}\right]_0^2 + i\left[4y - \frac{y^3}{3} + 2iy^2\right]_0^1$$

$$= \frac{8}{3} + i\left(4 - \frac{1}{3} + 2i\right)$$

$$= \frac{8}{3} - 2 + \frac{11i}{3}$$

$$= \frac{2}{3} + \frac{11}{3}i$$



Example 5.11

Find the value of $\int\limits_C (z^2+2z+1)dz$ where C is the circle $|z| = 1$. **(AU 2009)**

Solution:

By Cauchy's Integral theorem

$$f(z) = z^2 + 2z + 1 \qquad \int\limits_C f(z)\, dz = 0$$

$$\therefore \int\limits_C (z^2+2z+1)\, dz = 0$$

Example 5.12

Using Cauchy's integral formula, find the value of $\int\limits_C \frac{z+4}{z^2+2z+5}$, where C is the circle $|z+1-i| = 2$. **(AU 2008, 2012)**

Solution:

Given $C : |z+1-i| = 2$, this represents a circle with center at $-1+i$ and radius 2 units.

$$\therefore \quad \int\limits_C \frac{z+4}{z^2+2z+5}dz = \int\limits_C \frac{z+4}{(z+1+2i)(z+1-2i)}dz$$

The point $z = -1+2i$ lies inside the circle C and the point $z = -1-2i$ lies outside the circle C.

Hence by Cauchy's integral formula, we have

$$f(a) = \frac{1}{2\pi i}\int\limits_C \frac{f(z)}{z-a}dz = \frac{1}{2\pi i}\int\limits_C \frac{\frac{z+4}{z+1+2i}}{[z-(-1+2i)]}dz$$

$$\text{Taking } a = -1+2i, \quad \text{and} \quad f(z) = \frac{z+4}{z+1+2i}$$

$$\therefore \quad \int\limits_C \frac{f(z)}{z-a}dz = 2\pi i f(a)$$

$$\int\limits_C \frac{z+4}{(z+1+2i)(z+1-2i)}dz = 2\pi i f(a)$$

$$= 2\pi i\left[\frac{a+4}{a+1+2i}\right]$$

$$= 2\pi i\left[\frac{-1+2i+4}{-1+2i+1+2i}\right]$$

$$= 2\pi i\left[\frac{2i+3}{4i}\right]$$

$$= \left(\frac{3+2i}{2}\right)\pi$$

Example 5.13

State Cauchy's integral formula. Evaluate $\int\limits_C \frac{3z^2+z}{z^2-1}\,dz$ where C is the circle $|z-1| = 1$ using the formula. **(AU 2009)**

Solution:

Cauchy's integral formula

If $f(z)$ is analytic within and on a simple closed curve C and z_0 is any point inside C, then $f(z_0) = \frac{1}{2\pi i}\int\limits_C \frac{f(z)}{z-z_0}\,dz$ the integration round C being taken in the positive sense.

Here $\quad f(z) = 3z^2 + 2$

Now $\quad \dfrac{1}{z^2-1} = \dfrac{1}{(z+1)(z-1)} = \dfrac{A}{z-1} + \dfrac{B}{z+1} = \dfrac{1}{2}\left(\dfrac{1}{z-1} - \dfrac{1}{z+1}\right)$

$$\left(\because\ A = \frac{1}{2} \quad B = -\frac{1}{2}\right)$$

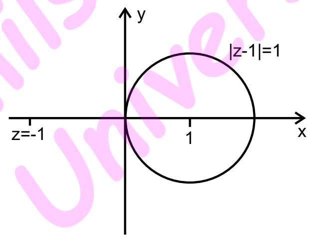
$$\int_C \frac{3z^2+z}{z^2-1}\,dz = \int_C \frac{3z^2+z}{(z+1)(z-1)}\,dz = \frac{1}{2}\int_C \left(\frac{1}{z-1} - \frac{1}{z+1}\right)(3z^2+z)\ \ dz$$

$$I = \frac{1}{2}\int_C \left(\frac{3z^2+z}{z-1} - \frac{3z^2+z}{z+1}\right)\ dz$$

Clearly $z = 1$ lies inside the circle and $z = -1$ lies outsides the circle $|z| = 1$.

$\therefore\ \dfrac{3z^2+z}{z^2+1}$ is analytic inside $|z| = 1$.

By Cauchy's integral theorem, we get



$$\int_{|z|=1} \frac{3z^2+z}{z^2-1}\,dz = 0$$

$$I = \frac{1}{2}\int_{|z-1|=1} \frac{3z^2+z}{z-1}\,dz$$

$$= \frac{1}{2}\cdot 2\pi i f(1), \quad \text{where } \ f(z) = 3z^2+z$$

$$= \pi i(3+1) = 4\pi i$$

Example 5.14

Using Cauchy's integral formula, evaluate $\displaystyle\int_C \frac{z}{(z-1)(z-2)^2}dz$, where C is the circle $|z-2| = \dfrac{1}{2}$. **(AU 2009)**

Solution:

Given $C : |z-2| = \dfrac{1}{2}$, this represents a circle with centre at 2 and radius $\dfrac{1}{2}$ unit.

$$\therefore \quad \int_C \frac{z}{(z-1)(z-2)^2}dz$$

The point $z = 1$ lines outside the circle C and the point $z = 2$ lies inside the circle C. Hence, by Cauchy's integral formula, we have

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz$$

take $n = 1$ or $f'(a) = \dfrac{1}{2\pi i} \displaystyle\int_C \frac{f(z)}{(z-a)^2} dt = \frac{1}{2\pi i} \int_C \frac{\frac{z}{z-1}}{(z-2)^2} dz$

here $a = 2$ and $f(z) = \dfrac{z}{z-1}$

$$\int_C \frac{f(z)}{(z-a)^2} dz = 2\pi i f'(a)$$

$$\int_C \frac{z}{(z-1)(z-2)^2} dz = 2\pi i f'(a)$$

$$= 2\pi i \left[\frac{d}{dz}\left(\frac{z}{z-1}\right)\right]_{z=2}$$

$$= 2\pi i \left[\frac{-1}{(z-1)^2}\right]_{z=2}$$

$$= -2\pi i$$

Example 5.15

Evaluate $\displaystyle\int_C \frac{z^2+1}{z^2-1} dz$ where C is a circle of unit radius and centre at

(i) $z = 1$ (ii) $z = i$. **(AU 2008)**

Solution:

(i) Given C is a circle with centre at $z = 1$ and radius 1 i.e. $|z-1| = 1$

$$\int_C \frac{z^2+1}{z^2-1} dz = \int_C \frac{z^2+1}{(z+1)(z-1)} dz$$

Hence the point $z = 1$ lies inside C and the point $z = -1$ lies outside C.

Hence by Cauchy's integral formula.

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz = \frac{1}{2\pi i} \int_C \frac{\frac{z^2+1}{z+1}}{z-1} dz$$

Here $a = 1$ and $f(z) = \dfrac{z^2+1}{z+1}$

$$\int_C \frac{z^2+1}{z^2-1}dz = \int_C \frac{z^2+1}{(z+1)(z-1)}dz = 2\pi i f(a)$$

$$= 2\pi i\left(\frac{a^2+1}{a+1}\right)$$

$$= 2\pi i\left(\frac{1+1}{1+1}\right)$$

$$= 2\pi i$$

(ii) Given C is a circle with centre at $z = i$ and radius 1 i.e. $|z - i| = 1$.

$$\therefore \quad \int_C \frac{z^2+1}{z^2-1}dz = \int_C \frac{z^2+1}{(z+1)(z-1)}dz$$

Here both the points $z = 1$ and $z = -1$ lie outside C.

Hence by Cauchy's theorem

$$\int_C \frac{z^2+1}{z^2-1}dz = 0$$

Example 5.16

Using Cauchy's integral formula evaluate $\displaystyle\int_C \frac{e^{2z}}{(z+1)^4}dz$ where C is the circle $|z| = 2$. **(AU 2009)**

Solution:

Given $C : |z| = 2$, which represents a circle with centre at zero and radius 2.

Hence $z = -1$ is a point inside C.

Here $a = -1$ and $f(z) = e^{2z}$

Hence by Cauchy's integral formula, where we have

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$$

Take $n = 3$

$$f'''(a) = \frac{3!}{2\pi i}\int_C \frac{f(z)}{(z-a)^4}dz$$

$$\therefore \quad \int_C \frac{e^{2z}}{(z+1)^4} = \frac{2\pi i}{3!}f'''(-1)$$

$$= \frac{\pi i}{3}\frac{d^3}{dz^3}\left[e^{2z}\right]_{z=-1}$$

$$= \frac{\pi i}{3}\left[8e^{2z}\right]_{z=-1}$$

$$= \frac{8\pi i}{3}e^{-2}$$

Example 5.17

Prove that $\frac{1}{2\pi i}\int_C \frac{z^3-z}{(z-z_0)^3}dz = 3z_0$, If C is a closed curves described in the positive sense and z_0 is inside C. What will be its value when z_0 lies outside C ?

Solution:

By Cauchy's integral formula

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$$

take $n = 2, a = z_0$ and $f(z) = z^3 - z$.

$$\therefore \quad f''(a) = \frac{2!}{2\pi i}\int_C \frac{z^3-z}{(z-z_0)^3}dz$$

$$= \frac{1}{2\pi i}\int \frac{z^3-z}{(z-z_0)^3}dz = \frac{1}{2}f''(z_0)$$

$$= \frac{1}{2}\frac{d^2}{dz^2}\left[z^3-z\right]_{z=z_0} = \frac{1}{2}(6z)_{z=z_0} = 3z_0$$

If z_0 lies outside C, then by Cauchy's theorem

$$\int_C \frac{z^3-z}{(z-z_0)^3}dz = 0$$

Example 5.18

Evaluate $\int_C \frac{dz}{2z-3}$ where c is the circle $|z| = 1$. **(AU 2010)**

Solution:

The Cauchy's integral formula is

$$\int_C \frac{f(z)}{z-z_0}\,dz = f(z_0)\cdot 2\pi \qquad (1)$$

Given $$\int \frac{dz}{2z-3} = \int \frac{1/2}{z-3/2}\,dz \qquad (2)$$

Comparing equation (2) with the L.H.S of equation (1) we get

$$f(z)=1 \qquad z_0=\frac{3}{2}$$

Here $z_0 = \frac{3}{2}$ = lies outside the circle $|z|$ =1.

$\therefore\ \frac{1}{2z-3}$ is analytic inside the circle C

$\therefore$ By Cauchy's integral theorem

$$\int_{|z|} \frac{dz}{2z-3} = 0$$

Example 5.19

What is the value of the integral $\int_C \left(\frac{3z^2+7z+1}{z+1}\right)\,dz$ where C is $|z| = \frac{1}{2}$?

(AU 2010)

Solution:

Since $z+1=0 \quad \Rightarrow \quad z=-1$

Here $1 > \frac{1}{2}$. $\therefore$ The point $z=-1$ is lies outside of the circle.

Hence $\int_C \left(\frac{3z^2+7z+1}{z+1}\right)\cdot dz = 0.$

Example 5.20

Evaluate $\oint_C \frac{z+4}{z^2+2z+5}dz$ where C is the circle $|z|=1$. **(AU 2009)**

Solution:

$|z| = 1$ is the circle whose center is (0, 0) and radius is 1.

$$\frac{z+4}{z^2+2z+5} = \frac{z+4}{(z+1+2i)(z+1-2i)}$$

The points $-1-2i$ and $-1+2i$ are lies outside of the circle $|z| = 1$.

By Cauchy's integral formula,

$$\begin{aligned}\int_C \frac{z+4}{z^2+2z+5} &= 2\pi i \times 0 \\ &= 0\end{aligned}$$

Example 5.21

Evaluate $\displaystyle\int_C \frac{dz}{(z-1)(z-2)(z-3)}$, where C is $|z| = 4$ **(AU 2009)**

Solution:

Given $C : |z| = 4$ which represents a circle with centre at 0 and radius 4 units.

Here $z = 1, z = 2$ and $z = 3$ all the three points lies inside C.

$$\frac{1}{(z-1)(z-2)(z-3)} = \frac{1/2}{z-1} + \frac{-1}{z-2} + \frac{1/2}{z-3}$$

(by splitting into partial fractions)

using Cauchy's integral formula, we have

$$\begin{aligned}\int_C \frac{dz}{(z-1)(z-2)(z-3)} &= \frac{1}{2}\int_C \frac{dz}{z-1} - \int_C \frac{dz}{z-2} + \frac{1}{2}\int_C \frac{dz}{z-3} \\ &= \frac{1}{2} 2\pi i f(1) - 2\pi i f(2) + \frac{1}{2}\cdot 2\pi i f(3) \\ &= \pi i - 2\pi i + \pi i \\ &= 0\end{aligned}$$

Example 5.22

If $f(a) = \displaystyle\int_C \frac{3z^2+7z+1}{z-a} dz$ where C is $|z| = \sqrt{2}$ find $f(3), f(1-i), f'(1-i)$ and $f''(1-i)$

Solution:

Given $C : |z| = \sqrt{2}$, which represents a circle with centre at 0 and radius 2 units.

(i) $f(3) = \int_C \frac{3z^2+7z+1}{z-3} dz$

The point $z = 3$ lies outside C

$\therefore$ by Cauchy's theorem

$$f(3) = \int_C \frac{3z^2+7z+1}{z-3} dz = 0$$

(ii) By Cauchy's integral formula, we have,

$$\therefore \quad f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)} dz$$

$$\therefore \quad f(1-i) = \int_C \frac{3z^2+7z+1}{z-(1-i)} dz$$

$$= 2\pi i f(a) \qquad \text{here } a = 1-i \ , \quad f(z) = 3z^2+7z+1$$

$$= 2\pi i(3a^2+7a+1)$$

$$= 2\pi i(3(1-i)^2+7(1-i)+1) = 2\pi(13-14i)$$

(iii) Given $f(a) = \int_C \frac{3z^2+7z+1}{z-a} dz$ differentiating with respect to a, we have

$$f'(a) = \int_C \frac{3z^2+7z+1}{(z-a)^2} dz$$

$z = 1-i$ lies inside $C \qquad \because \ |z| = \sqrt{2} < 2$

$$\therefore \quad f'(1-i) = \int_C \frac{3z^2+7z+1}{[z-(1-i)]^2}$$

$$= -2\pi i \frac{d}{dz}[3z^2+7z+1]_{z=1-i}$$

$$= -2\pi i[6z+7]_{z=+1-i}$$

$$= -2\pi i(13-6i)$$

$$= -2\pi(6+13i)$$

(iv) Now Differentiating $f'(a)$ again with respect to a

$$f''(a) = \int_C \frac{3z^2 + 7z + 1}{(z-a)^3} dz$$

$$f''(1-i) = \int_C \frac{3z^2 + 7z + 1}{(z-(1-i))^3} dz$$

$$= 2\pi i \, f''[3z^2 + 7z + 1]_{z=1-i}$$

$$= 2\pi i(6)$$

$$= 12\pi i$$

Example 5.23

Evaluate $\int_C \frac{3z^2 + 7z + 1}{z+1} dz$ where C is $|z| = \frac{1}{2}$. **(AU 2013)**

Solution:

$\begin{array}{r} z + 1 = 0 \\ z = -1 \end{array}$, which lies outside of $|z| = \frac{1}{2}$.

Hence by Cauchy's theorem $\int_C \frac{3z^2 + 7z + 1}{z+1} dz = 0$

Example 5.24

Evaluate $\displaystyle\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$, where C is the circle $|z| = 3$

(AU 2007, 2009, 2011)

Solution:

Given $C : |z| = 3$, which represents a circle with centre at 0 and radius 3 units.

Here the points $z = 1$ and $z = 2$ both the inside C.

Hence $a = 1$ and $a = 2$, $f(z) = \sin \pi z^2 + \cos \pi z^2$

By Cauchy's integral formula, we have

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)} dz + \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)} dz$$

(by partial fractions)

$$= -2\pi i\ f(1) + 2\pi i\ f(2)$$
$$= -2\pi i[\sin\pi + \cos\pi] + 2\pi i[\sin 4\pi + \cos 4\pi]$$
$$= -2\pi i(-1) + 2\pi i(1)$$
$$= 4\pi i$$

Example 5.25

Using Cauchy's integral formula evaluate $\int\limits_C \frac{e^z}{(z+2)(z+1)^2}dz$ where C is $|z| = 3$ **(AU 2008)**

Solution:

Given $|z| = 3$, which represents a circle with centre at 0 and radius 3 units. Hence the points $z = -1$ and $z = -2$ both lie inside the circle C.

Let $f(z) = e^z$.

$$\therefore \quad \int\limits_C \frac{e^z}{(z+2)(z+1)^2}dz = \int\limits_C \frac{e^z}{z+2}dz - \int\limits_C \frac{e^z}{z+1}dz + \int\limits_C \frac{e^z}{(z+1)^2}dz$$

(by partial fractions)

$$= 2\pi i f(-2) - 2\pi i f(-1) + \frac{2\pi i}{1!}f'(-1)$$
$$= 2\pi i[e^{-2}] - 2\pi i[e^{-1}] + 2ie^{-1}\pi$$
$$= 2\pi i e^{-2}$$

Example 5.26

Show that when $f(z)$ is analytic within and on a simple closed curve C and z_0 is not on C then $\int\limits_C \frac{f'(z)}{z-z_0}dz = \int\limits_C \frac{f(z)}{(z-z_0)^2}dz$

Solution:

(i) Suppose z_0 is an exterior point i.e. z_0 lies outside C, then both $\frac{f(z)}{(z-z_0)^2}$ and $\frac{f'(z)}{z-z_0}$ are analytic inside and on C

$\therefore$ by Cauchy's theorem

$$\int\limits_C \frac{f'(z)}{z-z_0}dz = \int\limits_C \frac{f(z)}{(z-z_0)^2} = 0$$

(ii) Suppose the point $z = z_0$ lies within C then by Cauchy's integral formula,

$$\int_C \frac{f'(z)}{z - z_0} dz = 2\pi i f'(z_0)$$

and by higher derivative formula

$$\int_C \frac{f(z)}{(z - z_0)^2} dz = 2\pi i f'(z_0)$$

$$\text{Hence} \quad \int_C \frac{f'(z)}{(z - z_0)} dz = \int_C \frac{f(z)}{(z - z_0)^2} dz$$

Example 5.27

Evaluate $\displaystyle\int_C \frac{\sin 2z}{(z - \pi i/4)^4}$, where C is $|z| = 1$. **(AU 2008)**

Solution:

Given $|z| = 1$, which represents a circle with centre at 0 and radius 1.

Hence $z = \dfrac{\pi i}{4}$ lies inside the circle C and $f(z) = \sin 2z$.

By Cauchy's integral formula, we have $f''(a) = \dfrac{n!}{2\pi i} \displaystyle\int_c \frac{f(z)}{(z - a)^{n+1}} dz$

take $n = 3$

$$\therefore \int_C \frac{f(z)}{(z - a)^4} = \frac{2\pi i}{3!} f'''(a)$$

$$\int_C \frac{\sin 2z}{(z - \pi i/4)^4} = \frac{\pi i}{3} \frac{d^3}{dz^3} [\sin 2z]_{z=\pi i/4}$$

$$= \frac{\pi i}{3} [-8 \cos 2z]_{z=\pi i/4}$$

$$= \frac{-8\pi i}{3} \cosh(\pi/2)$$

Example 5.28

Using Cauchy's integral formula evaluate $\displaystyle\int_C \frac{z + 1}{z^3 - 2z^2} dz$ where C is the circle $|z - 2 - i| = 2$ **(AU 2010)**

Solution:

Given $C : |z - 2 - i| = 2$, which represents a circle with centre at $z = 2 + i$ and radius 2.

Hence $z = 2$ lies inside the circle C and $z = 0$ lies outside the circle C.

$$\therefore \quad a = 2, f(z) = \frac{z+1}{z^2}$$

by Cauchy's integral formula we have

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)} dz$$

$$\text{or} \quad \int_C \frac{z+1}{z^2(z-2)} dz = 2\pi i f(a)$$

$$= 2\pi i \left(\frac{a+1}{a^2} \right)$$

$$= 2\pi i \left(\frac{2+1}{4} \right) = \frac{3\pi i}{2}$$

Example 5.29

Evaluate $\int_C \frac{z}{(z-1)(z+2)^2} dz$, where c is the circle $|z-2| = 3/2$, using Cauchy's integral formula. **(AU 2008)**

Solution:

Given $|z - 2| = 3/2$, which represents a circle with centre at $z = 2$ and radius 3/2. Hence $z = 1$ lies inside the circle C and $z = -2$ lies outside the circle C.

$$\text{Here} \quad a = 1 \quad \text{and} \quad f(z) = \frac{z}{(z+2)^2}$$

By Cauchy's integral formula, we have

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$$

$$\therefore \quad \int_C \frac{z}{(z-1)(z+2)^2} dz = 2\pi i f(a)$$

$$= 2\pi i \left(\frac{a}{(a+2)^2} \right)$$

$$= 2\pi i \left(\frac{1}{9} \right) \qquad (\because \ a = 1)$$

$$= \frac{2\pi i}{9}$$

Example 5.30

Evaluate $\int\limits_C \frac{z^3 + z + 1}{z^2 - 7z + 6} dz$ where C is the ellipse $4x^2 + 9y^2 + 1$, using Cauchy's integral formula.

Solution:

Given $4x^2 + 9y^2 + 1$ or $\frac{x^2}{(1/2)^2} + \frac{y^2}{(1/3)^2} = 1$ is the standard ellipse.

$$\therefore \qquad \int\limits_C \frac{z^3 + z + 1}{z^2 - 7z + 6} dz = \int\limits_C \frac{z^3 + z + 1}{(z-6)(z-1)} dz$$

Here both the points $z = 6$ and $z = 1$ lies outside the ellipse hence $\frac{z^3 + z + 1}{z^2 - 7z + 6}$ is analytic within and on C.

Hence by Cauchy's theorem $\int\limits_c \frac{z^3 + z + 1}{z^2 - 7z + 6} dz = .0$

Exercise 5(a)

☞ **Part - A**

1. State Cauchy's integral theorem.
2. State Cauchy's integral formula. **(AU 2007)**
3. State Cauchy's extended integral theorem for a multiply connected region.
4. State Cauchy's extended integral formula for a multiply connected region.
5. State Cauchy's integral formula for the n^{th} derivative of $f(z)$.
6. Define simply and multiply connected regions. **(AU 2008)**

7. Evaluate $\int_C z^2 dz$, where C is the curve $y = x^2$ passing through the points $(1,1)$ to $(2,4)$.

8. Evaluate $\int_C z^2 dz$, where C is straight line passing through the points $(1,1)$ & $(2,4)$.

9. Evaluate $\int_C \log z dz$, where C is the unit circle $|z| = 1$.

10. Evaluate $\int_C |z|\, dz$, where C is a semi-circle of the unit circle $|z| = 1$ from $z = -i$ to $z = i$.

11. Evaluate $\int_C \frac{1}{z} dz$ where C is the semi circular arc $|z| = 1$ above the real axis.

12. Evaluate $\int (x^2 - y^2) dz$ along the straight line from $(0,0)$ to $(0,1)$ and from $(0,1)$ to $(2,1)$

13. Show that $\int_C \log z dz = 4\pi i$, where C is the circle $|z| = 2$.

14. Show that $\int_0^{1+i} (x^2 + iy) dz = -\frac{1}{6} + i\frac{5}{6}$ along the parabola $y = x^2$.

15. Show that $\int_0^{1+i} (x^2 + iy) dz = -\frac{1}{6} + i\frac{13}{15}$ along the parabola $x = y^2$

16. Show that $\int_C \bar{z}^2 dt = 4\pi i$, where C is the circle $|z - 1| = 1$

17. Evaluate $\int_0^{1+i} |z|^2 dz$ along the line $3y = x$.

18. Evaluate $\int_C \frac{dz}{z^2(z^2+4)}$ where C is $|z| = 3$.

19. Evaluate $\int_C \frac{dz}{z-3}$ where C is the circle $|z-2| = 5$. **(AU 2009)**

20. Evaluate $\int_C \frac{e^z}{z} dz$ where C is the unit circle $|z| = 1$

☞ **Part - B**

21. Evaluate $\int_C (x+2y)dx + (y-2x)dy$ where C is the ellipse defined by $x = 4\cos\theta, y = 3\sin\theta$ and C is described in the anti-clock wise direction.

22. Evaluate $\int_{(0,1)}^{(2,5)} (3x+y)dx + (2y-x)dy$ along

    (i) The curve $y = x^2 + 1$

    (ii) The straight line joining $(0, 1)$ and $(2, 5)$

    (iii) The straight line from $(0, 1)$ to $(2, 1)$ and then from $(2, 1)$ to $(2, 5)$.

23. Evaluate $\int_{(0)}^{(1+i)} (x - y + ix^2)\, dz$ along the line from $z = 0$ to $z = 1+i$

24. Evaluate $\int_C (z^2+1)^2 dz$ along the arc of the cycloid $x = a(\theta - \sin\theta)$ $y = \theta(1 - \cos\theta)$ from the point $\theta = 0$ to $\theta = 2\pi$

25. Evaluate $\int_C \frac{dz}{z-2-i}$, where C is the boundary of the square bounded by the real and imaginary axes and the line $x = 1$ and $y = 1$.

26. Evaluate the following using Cauchy's integral formula

    (a) $\int_C \frac{z^3+1}{z^2-3iz}$, where C is $|z| = 1$

(b) $\int_C \frac{dz}{(z^2+4)^2} dz$ where C is $|z-2-i|=2$

(c) $\int_C \frac{z+1}{z^3-2z^2} dz$, where C is $|z-2-i|=2$

(d) $\int_C \frac{7z-1}{z^2-3z-4} dz$, where C is the ellipse $x^2+4y^2=4$

(e) $\int_C \frac{\tan z/2}{(z-a)^2} dz, -2 < a, < 2$, where C is the boundary of the square where sides lie along $x = \pm 2$ and $y = \pm 2$ described in the positive sense.

(f) $\int_C \frac{ze^z}{(z-a)^3} dz$ where $z=a$ lies inside the closed curve C.

(AU 2007)

(g) If $f(a) = \int_C \frac{4z^2+z+5}{(z-a)^3} dz$, where C is $|z| = 2$, find the values of $f(1), f(i), f'(-1)$ and $f''(-i)$

(h) $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z+1)(z+2)} dz$, where $C = |z| = 3$.

(i) $\int_C \frac{\sinh 2z}{z^4} dz$, where C is the boundary of the square $x = \pm 2$ and $y = \pm 2$

(j) $\int_C \frac{z dz}{(9-z^2)(z+1)}$, where C is $|z|=2$ **(AU 2008)**

(k) $\int_C \frac{e^{3z} dz}{z-\pi i}$, where C is $|z-1|=4$

(l) $\int_C \frac{\sin 3z}{z+\pi/2} dz$, where C is $|z|=5$

(m) $\int_C \frac{\sin z}{z^2 e^z} dz$, where C is $|z|=1$

(n) $\int_C \frac{\sin^6 z dz}{(z-\pi/6)^3}$, where C is $|z|=1$

(o) $\int_c \frac{dz}{(z^2+4)^3}$, where C is the circle $|z-i|=2$ **(AU 2009)**

27. Show that $\int_C \frac{4-3z}{z(z-1)(z-2)} dz = 2\pi i$, where C is the circle $|z|=3/2$. **(AU 2010)**

28. Prove that $\frac{1}{2\pi i}\int_C \frac{e^{zt}}{(z^2+1)^2} dz = \frac{t\sin t}{2}$ if $t>0$ and is the circle $|z|=3$.

29. If C is the circle $|z| = 3$ described in the positive sense and if $g(z_0) = \int_C \frac{2z^2-z-2}{z-z_0} dz, |z| \neq 3$, show that $g(2) = 8\pi i$. What is the value of $g(z_0)$ when $|z_0| > 3$?

30. If $f(a) = \int_C \frac{3z^2+6z+1}{z-a} dz$, where C is the circle $x^2+y^2=4$, find the value of $f(3), f(1-i)$ and $f''(i)$.

31. If C is a closed curve described in the positive sense and $\phi(z_0) = \int_C \frac{z^4+z}{z-z_0} dz$ show that $\phi(z_0) = 12\pi i z_0^2$, when z_0 is inside C and $\phi(z_0)=0$ when z_0 lies outside C.

5.4 Taylor and Laurent's Expansions

Let us consider the infinite series $f_1(z)+f_2(z)+f_3(z)+\cdots$, whose terms are functions of the complex variable z.

Let $S_n(z)$ denote the sum of the first n terms of the series. If $S_n(z)$ tends to a limit $S(z)$ as $n\to\infty$ for all z in a region R, then the series as said to converge or to be convergent in the region R and to have a sum $S(z)$.

$$\text{i.e.,}\quad \lim_{n\to\infty} S(z) - S_n(z) = 0$$

A series which is not convergent is said to diverge or to be divergent.

Every analytic function has a power series representation called the *Taylor series*. Analytic function can also be represented by another type of series called

■ **Note :**

1. $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$ if $|x| < 1$.

   $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$ if $|x| < 1$.

   $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$ if $|x| < 1$.

   $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$ if $|x| < 1$.

2. $(1+z)^{-1} = 1 - z + z^2 - z^3 + \cdots$, Region of validity is $|z| < 1$.

3. $\left(1+\frac{1}{z}\right)^{-1} = 1 - \frac{1}{z} + \frac{1}{z^2}\cdots$, Region of validity is $\left|\frac{1}{z}\right| < 1 \Rightarrow |z| > 1$

4. If we have only positive powers of (z-a) then the series is Taylor's series about the point $z = a$, if we have positive and negative powers of $(z-a)$, then the series is a Laurent's series.

5. The part $\sum_{n=0}^{\infty} a_n(z-a)^n$, consisting of positive integral powers of $(z-a)$ is called the analytic part of the Laurent's series, and $\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$, consisting of negative integral powers of $(z-a)$ is called the principal part of the Laurent's series.

Example 5.31

Expand $f(z) = \sin z$ in a Taylor series about $z = \pi/4$ and determine the region of convergence of this series. **(AU 2006, 2009)**

Solution:

$$\text{Given } f(z) = \sin z \qquad \therefore \quad f(\pi/4) = \frac{1}{\sqrt{2}}$$

$$f'(z) = \cos z \qquad \therefore \quad f'(\pi/4) = \frac{1}{\sqrt{2}}$$

$$f''(z) = -\sin z \qquad \therefore \quad f''(\pi/4) = -\frac{1}{\sqrt{2}}$$

$$f'''(z) = -\cos z \qquad \therefore \quad f'''(\pi/4) = -\frac{1}{\sqrt{2}}$$

The Taylor's series of $f(z) = \sin z$ at $z = \pi/4$, is given by

$$f(z) = f(\pi/4) + \frac{(z-\pi/4)}{1!} f'(\pi/4) + \frac{(z-\pi/4)^2}{2!} f''(\pi/4) + \cdots$$

$$\therefore \quad \sin z = \frac{1}{\sqrt{2}} + \frac{(z-\pi/4)}{1!}\left(\frac{1}{\sqrt{2}}\right) + \frac{(z-\pi/4)^2}{2!} - \left(\frac{1}{\sqrt{2}}\right) + \frac{(z-\pi/4)^3}{3!} - \left(\frac{1}{\sqrt{2}}\right) + \cdots$$

$$= \frac{1}{\sqrt{2}}\left[1 + \frac{(z-\pi/4)}{1!} - \frac{(z-\pi/4)^2}{2!} - \frac{(z-\pi/4)^3}{3!} + \cdots\right]$$

The region of convergence is $|z - \pi/4| < \infty$.

Example 5.32

Expand $\cos z$ into a Taylor's series about the point $z = \pi/2$ and determine the region of convergence. **(AU 2007)**

Solution:

Let $f(z) = \cos z$ at $\pi/2$ $\quad f(\pi/2) = 0$

$f'(z) = -\sin z \quad f'(\pi/2) = -1$

$f''(z) = -\cos z \quad f''(\pi/2) = 0$

$f'''(z) = \sin z \quad f'''(\pi/2) = 1$

$\therefore$ The Taylors series for $\cos z$ at $z = \pi/2$ is

$$\cos z = f(\pi/2) + \frac{(z-\pi/2)}{1!} f'(\pi/2) + \frac{(z-\pi/2)^2}{1!} f''(\pi/2) + \cdots$$

$$= \frac{-(z-\pi/2)}{1!} + \frac{(z-\pi/2)^3}{3!} - \frac{(z-\pi/2)^5}{5!} + \cdots$$

The region of convergence is $|z - \pi/2| < \infty$.

Example 5.33

Obtain the Taylor series of $f(z) = \dfrac{1-z}{z^2}$ in power of $(z-1)$.

Solution:

$f(z)$ is not analytic at $z = 0$, hence we consider a circle with at $z = 1$ and radius 1, such that $z = 0$ is excluded from the region.

Then the region of validity for Taylor's series will be $|z-1| < 1$

$$\begin{aligned} f(z) &= \frac{1-z}{z^2} & \therefore\ f(1) &= 0 \\ f'(z) &= \frac{-2}{z^3} + \frac{1}{z^2} & f'(1) &= -1 \\ f''(z) &= \frac{6}{z^4} - \frac{2}{z^3} & f''(1) &= 4 \\ f'''(z) &= -\frac{24}{z^5} + \frac{6}{z^4} & f'''(1) &= -18 \end{aligned}$$

Hence the required Taylor series expansion is,

$$f(z) = f(1) + \frac{(z-1)}{1!} f'(1) + \frac{(z-1)^2}{2!} f''(1) + \cdots$$
$$f(z) = \frac{1-z}{z^2} = -(z-1) + 2(z-1)^2 - 3(z-1)^3 + 4(z-1)^4 \cdots$$

Example 5.34

Expand ze^{2z} in a Taylor's series about $z = -1$ and determine the region of convergence.

Solution:

$$\begin{aligned} f(z) &= ze^{2z} = ze^{2(z+1)}e^{-2} \\ &= \frac{1}{e^2}\left[(z+1)e^{2(z+1)} - e^{2(z+1)}\right] \\ &= \frac{1}{e^2}\Bigg\{(z+1)\left[1 + \frac{2(z+1)}{1!} + \frac{4(z+1)^2}{2!} + \cdots\right] \\ &\qquad - \left[1 + \frac{2(z+1)}{1!} + \frac{4(z+1)^2}{2!} + \cdots\right]\Bigg\} \\ &= \frac{1}{e^2}\left[\left((z+1) + \frac{2(z+1)^2}{1!} + \frac{2^2(z+1)^2}{2!} + \cdots\right)\right. \\ &\qquad \left. - \left(1 + \frac{2(z+1)}{1!} + \frac{2^2(z+1)^2}{2!} + \cdots\right)\right] \\ &= \frac{1}{e^2}\left[1 + \left(1 - \frac{2}{1!}\right)(z+1) + \left(\frac{2}{1!} - \frac{2^2}{2!}\right)(z+1)^2\right. \\ &\qquad \left. + \left(\frac{2^2}{2!} - \frac{2^3}{3!}\right)(z+1)^3 + \cdots\right] \end{aligned}$$

Example 5.35

Find the Taylors series expansion of $f(z) = \dfrac{z^2-1}{(z+2)(z+3)}$ in $|z| < 2$.

Solution:

Given

$$\begin{aligned}
f(z) &= \frac{z^2-1}{(z+2)(z+3)}\\
&= 1 + \frac{3}{z+2} - \frac{8}{z+3} \qquad \text{(by partial fractions)}\\
&= 1 + \frac{3}{2\left(1+\frac{z}{2}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}\\
&= 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}\\
&= 1 + \frac{3}{2}\left(1 - \frac{z}{2} + \frac{z^2}{2^2} - \frac{z^3}{2^3} + \cdots\right) - \frac{8}{3}\left(1 - \frac{z}{3} + \frac{z^2}{3^2} - \frac{z^3}{3^3} + \cdots\right)\\
&= \left(1 + \frac{3}{2} - \frac{8}{3}\right) + \left(-\frac{3}{2^2} + \frac{8}{3^2}\right)z + \left(\frac{3}{2.2^2} - \frac{8}{3.3^2}\right)z^2 + \cdots\\
&= -\frac{1}{6} + \left(\frac{8}{3^2} - \frac{3}{2^2}\right)z - \left(\frac{8}{3^3} - \frac{3}{2^3}\right)z^2 + \cdots
\end{aligned}$$

Example 5.36

Obtain Taylor's series for $f(z) = \dfrac{1}{(z+2)(1+z^2)}$ in $|z| < 1$

Solution:

Given $f(z) = \dfrac{1}{(z+2)(1+z^2)}$

$$= \frac{1}{5}\left[\frac{1}{z+2} + \frac{2-z}{1+z^2}\right] \qquad \text{by splitting into partial fractions}$$

For $|z| < 1$

$$\begin{aligned}
f(z) &= \frac{1}{10}\frac{1}{\left(1+\frac{z}{2}\right)} + \frac{1}{5}(2-z)\frac{1}{(1+z^2)}\\
&= \frac{1}{10}\left(1+\frac{z}{2}\right)^{-1} + \frac{1}{5}(2-z)(1+z^2)^{-1}\\
&= \frac{1}{10}\left[1 - \frac{z}{2} + \frac{z^2}{4}\cdots\right] + \frac{(2-z)}{5}[1 - z^2 + z^4 - z^6 + \cdots]\\
&= \frac{1}{10}\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{2^n} + \frac{2-z}{5}\sum_{n=0}^{\infty}(-1)^n z^{2n}
\end{aligned}$$

Example 5.37

Expand $f(z) = \dfrac{1}{z^2 - 3z + 2}$ in Laurent's series valid in the region $1 < |z| < 2$.

(AU 2009, 2010)

Solution:

Let $f(z) = \dfrac{1}{z^2 - 3z + 2} = \dfrac{1}{(z-2)(z-1)} = \dfrac{1}{z-1} - \dfrac{1}{z-2}$

$f(z)$ is an analytic function in the region $1 < |z| < 2$

$$\text{Hence} \qquad f(z) = \frac{1}{-2(1 - \frac{z}{2})} - \frac{1}{z(1 - \frac{1}{z})}$$

$$= -\frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1}$$

The expansion of $\left[1 - \dfrac{z}{2}\right]^{-1}$ is valid when $\left|\dfrac{z}{2}\right| < 1$ i.e., $|z| < 2$ and the expansion of $\left(1 - \dfrac{1}{z}\right)^{-1}$ is valid when $\left|\dfrac{1}{z}\right| < 1$ i.e., $|z| > 1$

Hence in the annular region $1 < |z| < 2$, $f(z)$ can be expanded in Laurent's series as

$$f(z) = -\frac{1}{2}\left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \cdots\right] - \frac{1}{z}\left[1 + \frac{1}{2} + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \cdots\right]$$

$$= -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} - \sum_{n=1}^{\infty} \frac{1}{z^{n+1}}$$

Example 5.38

Expand $f(z) = \dfrac{1}{z^2 - 3z + 2}$ as Laurent's expansion in the region $0 < |z - 1| < 1$ and obtain its residue at $z = 1$. **(AU 2009)**

Solution:

$$f(z) = \frac{1}{z^2 - 3z + 2}$$

$$\Rightarrow \frac{1}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$$

$$\Rightarrow 1 = A(z-2) + B(z-1)$$

Put $z = 1$ we get	Put $z = 2$ we get
$1 = -A + 0$	$1 = 0 + B$
$\therefore\ \boxed{A = -1}$	$\therefore\ \boxed{B = 1}$

$$\therefore \qquad f(z) = -\frac{1}{z-1} + \frac{1}{z-2}$$

$$0 < |z-1| < 1$$

Let $\quad u = z - 1 \quad \therefore \quad z = u + 1 \quad$ (i.e.,) $\quad |u| < 1$

$$\begin{aligned} f(z) &= -\frac{1}{u} + \frac{1}{u-1} \\ &= -\frac{1}{u} - \frac{1}{(1-u)} \\ &= \frac{-1}{u} - (1-u)^{-1} \\ &= \frac{-1}{u} - (1 + u - u^2 + \ldots\ldots) \\ &= \frac{-1}{z-1} - 1 - (z-1) - (z-1)^2 + \ldots\ldots \\ &= -\left[\frac{-1}{z-1} + 1 + (z-1) + (z-1)^2 + \ldots\ldots\right] \end{aligned}$$

Residues

$$f(z) = \frac{1}{(z-1)(z-2)}$$

Here $z = 1$ is a pole of order 1

$z = 2$ is a pole of order 1

$$\text{Res } [z = +1] = \underset{z \to 1}{Lt}(z-1) \cdot f(z) = \underset{z \to 1}{Lt}(z-1) \cdot \frac{1}{(z-1)(z-2)} = -1$$

$$\text{Res } [z = 2] = \underset{z \to 2}{Lt}(z-2) \cdot f(z) = \underset{z \to 2}{Lt}(z-2) \cdot \frac{1}{(z-1)(z-2)} = 1$$

$\therefore\ $ Res $[z = 1] = -1$

Example 5.39

Find the Laurent's expansion for $f(z) = \dfrac{1}{z^2(1-z)}$ for the region

(i) $0 < |z| < 1$ (ii) $1 \leq |z| < 4$ **(AU 2010)**

Solution:

(i) The region $0 < |z| < 1$ is the interior of the circle $|z| = 1$, with the exception of the point $z = 0$.

$$\therefore \quad f(z) = \frac{1}{z^2(1-z)} = \frac{1}{z^2}(1-z)^{-1}$$
$$= \frac{1}{z^2}[1 + z + z^2 + z^3 + \cdots]$$
$$= \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 \cdots$$

is the required expansion in $0 < |z| < 1$

(ii) $$f(z) = \frac{1}{z^2(1-z)} = \frac{1}{-z^3\left(1 - \frac{1}{z}\right)} = -\frac{1}{z^3}\left(1 - \frac{1}{z}\right)^{-1}$$
$$= -\frac{1}{z^3}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} \cdots\right]$$
$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+3}}, \qquad \text{binomial expansion is valid if } |z| > 1$$

and hence this is the required expansion in $1 < |z| < 4$.

■ **Note :** In fact it is valid in the region $1 < |z| < M$, where M is any large positive number.

Example 5.40

Find the expansion of $f(z) = \frac{1}{z - z^3}$ in the region $1 < |z - 1| < 2$ **(AU 2009)**

Solution:

$$\text{Let} \quad f(z) = \frac{1}{z - z^3} = \frac{1}{z(1 - z^2)} = \frac{1}{z(1+z)(1-z)}$$
$$= \frac{1}{z} + \frac{-1/2}{z+1} + \frac{1/2}{1-z}$$

Put $z - 1 = u$

$\therefore \quad z = u + 1$

$$\therefore \quad f(z) = \frac{1}{u+1} + \frac{-1/2}{u+2} - \frac{1/2}{u}$$
$$= \frac{1}{u+1} - \frac{1}{2}\left(\frac{1}{u+2}\right) - \frac{1}{2}\left(\frac{1}{u}\right)$$

The region $1 < |z-1| < 2$ is equivalent to $1 < |u| < 2$

$$\therefore \qquad f(z) = \frac{1}{u\left(1+\frac{1}{u}\right)} + \frac{1}{4}\frac{1}{1+\frac{u}{2}} - \frac{1}{2u}$$
$$= \frac{1}{u}\left(1+\frac{1}{u}\right)^{-1} - \frac{1}{4}\left(1+\frac{u}{2}\right)^{-1} - \frac{1}{2u}$$

The first two terms can be expanded binomially since $1 < |u| < 2$

$$\therefore\ f(z) = \frac{1}{u}\left[1-\frac{1}{u}+\frac{1}{u^2}-\frac{1}{u^3}+\cdots\right]-\frac{1}{4}\left[1-\frac{u}{2}+\left(\frac{u}{2}\right)^2-\left(\frac{u}{2}\right)^3+\cdots\right]-\frac{1}{u}$$

$$= \frac{1}{z-1}\left[1-\frac{1}{z-1}+\frac{1}{(z-1)^2}-\frac{1}{(z-1)^3}+\cdots\right]$$
$$-\frac{1}{4}\left[1-\frac{z-1}{2}+\left(\frac{z-1}{2}\right)^2-\left(\frac{z-1}{2}\right)^3+\cdots\right]-\frac{1}{2(z-1)}$$

$$f(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(z-1)^{n+1}} - \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+2}}(z-1)^n - \frac{1}{2(z-1)}$$

Example 5.41

Expand $\dfrac{1}{(z+1)(z+3)}$ as a Laurent's series in the regions.

(i) $|z| < 1$ and

(ii) $1 < |z+1| < 2$. **(AU 2007, 2009)**

Solution:

Given $f(z) = \dfrac{1}{(z+1)(z+3)} = \dfrac{A}{z+1} + \dfrac{B}{z+3} = \dfrac{1/2}{z+1} - \dfrac{1/2}{z+3}$

(i) Given $|z| < 1$

$$1 = A(z+3) + B(z+1)$$
$$1 = 2A \quad \Rightarrow \quad A = \frac{1}{2}$$
$$1 = -2B \quad \Rightarrow \quad B = -\frac{1}{2}$$

$$\frac{1}{(z+1)(z+3)} = \frac{1}{2}\left\{\frac{1}{1+z} - \frac{1}{3\left(1+\frac{z}{3}\right)}\right\}$$

$$= \frac{1}{2}\left\{(1+z)^{-1} - \frac{1}{3}\left(1+\frac{z}{3}\right)^{-1}\right\}$$
$$= \frac{1}{2}\left\{(1 - z + z^2 - z^3 + \ldots) - \frac{1}{3}\left(1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \ldots\right)\right\}$$
$$= \frac{1}{2}\left\{1 - z + z^2 - z^3 + \ldots - \frac{1}{3} + \frac{z}{9} - \frac{z^2}{27} + \ldots\right\}$$
$$f(z) = \frac{1}{2}\left\{\frac{2}{3} - \frac{8z}{9} + \frac{26}{27}z^2 - \ldots\right\} = \frac{1}{3} - \frac{4z}{9} + \frac{13}{27}z^2 - \ldots$$

(ii) Put $z + 1 = u$

$\therefore \quad 0 < |z+1| < 2$ becomes $0 < |u| < 2$

Now, $$\frac{1}{(z+1)(z+3)} = \frac{1}{(z+1)(z+1+2)} = \frac{1}{u(u+2)}$$

Instead of expanding $\frac{1}{(z+1)(z+3)}$ in powers of $z+1$ it is enough to expand $\frac{1}{u(u+2)}$ in powers of u in $0 < |u| < 2$.

$$\frac{1}{u(u+2)} = \frac{1}{2u\left(1+\frac{u}{2}\right)}$$

$0 < |u| < 2$, we have $0 < |u|$ and $|u| < 2$

(i.e.,) $\quad \frac{|u|}{2} < 1 \quad$ (or) $\quad \left|\frac{u}{2}\right| < 1$

$$\frac{1}{u(u+2)} = \frac{1}{2u\left(1+\frac{u}{2}\right)} = \frac{1}{2u}\left(1+\frac{u}{2}\right)^{-1}$$
$$= \frac{1}{2u}\left[1 - \frac{u}{2} + \left(\frac{u}{2}\right)^2 - \left(\frac{u}{2}\right)^3 + \ldots\right]$$
$$\frac{1}{u(u+2)} = \frac{1}{2u} - \frac{1}{4} + \frac{1}{8}u - \frac{1}{16}u^2 + \ldots$$

Replacing u by $z+1$ we get

$$\frac{1}{(z+1)(z+3)} = \frac{1}{2(z+1)} - \frac{1}{4} + \frac{1}{8}(z+1) - \frac{1}{16}(z+1)^2 + \ldots$$

Example 5.42

If $f(z) = \frac{z+4}{(z+3)(z-1)^2}$, find Laurent's series expansions in

(i) $0 < |z-1| < 4$ (ii) $|z-1| > 4$ **(AU 2009)**

Solution:

Let $f(z) = \dfrac{z+4}{(z+3)(z-1)^2} = \dfrac{1}{16(z+3)} - \dfrac{1}{16(z-1)} + \dfrac{5}{4(z-1)^2}$

(i) $0 < |z-1| < 4$

put $z - 1 = u$

$$\therefore \quad f(z) = \frac{1}{16(u+4)} - \frac{1}{16(u)} + \frac{5}{4u^2}$$

$$= \frac{1}{64}\left(1+\frac{u}{4}\right)^{-1} - \frac{1}{16u} + \frac{5}{4u^2}$$

$$= \frac{1}{64(1+\frac{u}{4})^{-1}} - \frac{1}{16u} + \frac{5}{4u^2}$$

$$= \frac{1}{64}\left[1 - \frac{u}{4} + \left(\frac{u}{4}\right)^2 - \left(\frac{u}{4}\right)^3 + \cdots\right] - \frac{1}{16u} + \frac{5}{4u}$$

$$f(z) = \frac{1}{64}\left[1 - \frac{z-1}{4} + \left(\frac{z-1}{4}\right)^2 - \left(\frac{z-1}{4}\right)^3 + \cdots\right] - \frac{1}{16(z-1)} + \frac{5}{4(z-1)}$$

This is the required Laurent's series expansion for $f(z)$ in $0 < |z-1| < 4$.

(ii) $|z-1| > 4$ or $\left|\dfrac{4}{z-1}\right| < 1$

put $z - 1 = u$ or $\left|\dfrac{4}{u} < 1\right|$

$$\therefore \quad f(z) = \frac{1}{16(u+4)} - \frac{1}{16u} + \frac{5}{4u^2}$$

$$= \frac{1}{16u\,(1+\frac{4}{u})} - \frac{1}{16u} + \frac{5}{4u^2}$$

$$= \frac{1}{16u}\left(1+\frac{4}{u}\right)^{-1} - \frac{1}{16u} + \frac{5}{4u^2}$$

$$= \frac{1}{16u}\left[1 - \frac{4}{u} + \left(\frac{4}{u}\right)^2 + \left(\frac{4}{u}\right)^3 + \cdots\right] - \frac{1}{16u} + \frac{5}{4u^2}$$

$$= \frac{1}{16(z-1)}\left[1 - \frac{4}{z-1} + \left(\frac{4}{z-1}\right)^2 - \left(\frac{4}{z-1}\right)^3 + \cdots\right] - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2}$$

$$\therefore \quad f(z) = \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3} - \frac{4}{(z-1)^4} + \cdots$$

is the required Laurent's series expansion for $f(z)$ in $|z-1| > 4$.

Example 5.43

Expand the function $f(z) = \dfrac{z^2-1}{z^2+5z+6}$ in Laurent's series for $|z| > 3$.

(AU 2013)

Solution:

$$\frac{z^2-1}{z^2+52+6} = A + \frac{B}{z+2} + \frac{C}{z+3}$$
$$z^2 - 1 = A(z+2)(z+3) + B(z+3) + C(z+2)$$

Put $z = -2$, $B = 3$; Put $z = -3$, $C = -8$ Equate Coeff of z^2, $A = 1$

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

When $|z| > 3$

$$f(z) = 1 + \frac{3}{z\left(1+\frac{1}{z}\right)} - \frac{8}{z\left(1+\frac{3}{z}\right)}$$
$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$
$$= 1 - \frac{3}{z}\sum_{1}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{z}\sum_{1}^{\infty}(-1)^n\left(\frac{3}{z}\right)^n$$

Example 5.44

Find the Laurent's series of $f(z) = \dfrac{7z-2}{z(z+1)(z+2)}$ in $1 < |z+1| < 3$.

(AU 2010)

Solution:

Take $u = z+1 \Rightarrow z = u-1$

$$\frac{7z-2}{z(z+1)(z+2)} = \frac{7(u-1)-2}{(u-1)\cdot u(u-3)} = \frac{7u-9}{u(u+1)(u-3)}$$

Using Partial Function

$$\frac{7u-9}{u(u+1)(u-1)} = \frac{A}{u} + \frac{B}{u+1} + \frac{C}{u-3}$$

$$7u-9 = A(u+1)(u-3) + Bu(u-3) + Cu(u+1)$$

$$\text{Put } u=0 \Rightarrow -9=-3A \Rightarrow A=3$$

$$\text{Put } u=-1 \Rightarrow -16=4B \Rightarrow B=-4$$

$$\text{Put } u=3 \Rightarrow 12=12C \Rightarrow C=1$$

$$\therefore \quad \frac{7u-9}{u(u+1)(u-3)} = \frac{3}{u} - \frac{4}{u+1} + \frac{1}{u-3} \qquad \text{if } 1<|u|<3$$

$$= \frac{3}{u} - \frac{4}{u}\left(1+\frac{1}{u}\right)^{-1} + \frac{1}{-3}\left(1-\frac{u}{3}\right)^{-1}$$

$$= \frac{3}{u} - \frac{4}{u}\left[1-\frac{1}{u}+\left(\frac{1}{u}\right)^2 - \ldots\right]$$

$$+ \frac{1}{-3}\left[1+\frac{u}{3}+\left(\frac{u}{3}\right)^2 + \ldots\right]$$

The second term of the series is valid for $\left|\frac{1}{u}\right| < 1$ and the third term of the series is valid for $\left|\frac{u}{3}\right| < 1$ and finally put $u = z+1$. Hence the series is valid for $1 < |z+1| < u$.

Example 5.45

Find all possible Laurent's expansions of $f(z) = \dfrac{4-3z}{z(1-z)(2-z)}$ about $z=0$. Indicate the region of convergence in each case. **(AU 2009)**

Solution:

$$f(z) = \frac{4-3z}{z(1-z)(2-z)}$$

$$\frac{4-3z}{z(1-z)(2-z)} = \frac{A}{z} + \frac{B}{1-z} + \frac{C}{2-z}$$

$$A=2, \qquad B=1, \qquad C=1$$

$$\text{(i)} \quad \frac{4-3z}{z(1-z)(2-z)} = \frac{2}{z} + \frac{1}{1-z} + \frac{1}{2-z}$$

$$= \frac{2}{z} + (1-z)^{-1} + \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} \quad \text{valid in } |z|<1$$

$$= \frac{2}{z} + \left(1+z+z^2+z^3+\ldots\infty\right)$$

$$+ \frac{1}{2}\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\ldots\infty\right)$$

(ii) $$\frac{4-3z}{z(1-z)(2-z)} = \frac{2}{z} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} \text{ valid in } 1<|z|<2$$
$$= \frac{2}{z} - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n} + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^{n}$$

(iii) $$\frac{4-3z}{z(1-z)(2-z)} = \frac{2}{z} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} \text{ valid in } |z|>2$$
$$= \frac{2}{z} - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n} - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^{n}$$

Example 5.46

Expand $\frac{1}{z(z-1)}$ as Laurent's series

(i) about $z=0$ in powers of z

(ii) about $z=1$ in powers of $z-1$.

Also state the region of validity. **(AU 2009)**

Solution:

(i) The only points where $f(z)$ is not analytic are 0 and 1. Hence $f(z)$ can be expanded as a Laurent's series in the annulus $0<|z|<1$.

$$\begin{aligned} f(z) &= \frac{1}{z(z-1)} \\ &= -\frac{1}{z}(1-z)^{-1} \\ &= -\frac{1}{z}(1+z+z^2+z^3+\cdots) \qquad (\text{since } |z|<1) \\ &= -[1/z+1+z+z^2+z^3+\cdots] \end{aligned}$$

This is the Laurent's series expansion of $f(z)$ in $0<|z|<1$.

(ii) $f(z)$ is analytic in $0<|z-1|<1$ and hence can be expanded as a Laurent's series in powers of $z-1$ in this region.

put $z-1=u$.

$$\therefore \quad f(z) = \frac{1}{z(z-1)} = \frac{1}{(u+1)(u)} = \frac{1}{u}(1+u)^{-1}$$

$$= \frac{1}{u}[1 - u + u^2 - u^3 + u^4 \cdots]$$
$$= \frac{1}{u} - 1 + u - u^2 + \cdots$$
$$= \frac{1}{(z-1)} - 1 + (z-1) - (z-1)^2 + \cdots$$

This is the Laurent's series expansion in $0 < |z-1| < 1$.

Example 5.47

Find the Laurent's series of $f(z) = \dfrac{z^2-1}{z^2+5z+6}$ valid in the region $2 < |z| < 3$. **(AU 2009)**

Solution:

$$\begin{array}{r|l} & 1 \\ \hline z^2+5z+6 & z^2-1 \\ & z^2+5z+6 \\ \hline & -5z-7 \end{array}$$

$$f(z) = \frac{z^2-1}{z^2+5z+6} = 1 - \frac{5z-7}{z^2+5z+6}$$
$$= 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

Given $\quad 2 < |z| < 3$

$$\Rightarrow \quad |z| > 2 \text{ and } \quad |z| < 3$$
$$= 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$
$$= 1 + \frac{3}{z}\sum_{0}^{\infty}\left(\frac{2}{z}\right)^n(-1)^n - \frac{8}{3}\sum_{0}^{\infty}\left(\frac{z}{3}\right)^n(-1)^n$$
$$= 1 + \frac{3}{z}\sum_{0}^{\infty}(-1)^n\frac{2^n}{z^n} - \frac{8}{3}\sum_{0}^{\infty}(-1)^n\frac{z^n}{3^n}$$

Example 5.48

Obtain the expansion for $f(z) = \dfrac{z-1}{z^2}$ in powers of $(z-1)$. Indicate the region of validity.

Solution:

The function $f(z) = \dfrac{z-1}{z^2}$ is not analytic at $z = 0$. Hence $f(z)$ will be analytic in the annular region $0 < |z-1| < 1$ and $1 < |z-1| < \infty$.

(i) In the region $0 < |z-1| < 1$

$$f(z) = \frac{z-1}{z^2} = \frac{u}{(u+1)^2} = u(1+u)^{-2}$$
$$= u[1 - 2u + 3u^2 - 4u^3 + \cdots]$$
$$= (z-1)[1 - 2(z-1) + 3(z-1)^2 - 4(z-1)^3 + \cdots]$$

is the Laurent's series expansion in $0 < |z-1| < 1$.

(ii) In the region $1 < |z-1| < \infty$

$$f(z) = \frac{u}{(u+1)^2} = \frac{1}{u}\left[1 + \frac{1}{u}\right]^{-2}$$
$$= \frac{1}{u}\left[1 - \frac{2}{u} + \frac{3}{u^2} - \frac{4}{u^3} + \cdots\right]$$
$$= \frac{2}{u} - \frac{3}{u^2} + \frac{4}{u^3} - \frac{1}{u^4} + \cdots$$
$$= \frac{1}{z-1} - \frac{2}{(z-1)^2} + \frac{3}{(z-1)^3} - \frac{4}{(z-1)^4} + \cdots$$

is the Laurent's series valid in the region $|u| > 1$ or $1 < |z-1| < \infty$.

Example 5.49

Find all the Laurent's series expansion for $f(z) = \dfrac{1}{z(1-z)^2}$ and specify the regions in which those expansions are valid. **(AU 2008)**

Solution:

The function $f(z)$ is not analytic at $z = 0$ and $z = 1$. To expand by Laurent's series, we have to find an annular region bounded by two concentric circles which does not include $z = 0$ and $z = 1$.

(i) Consider the region $0 < |z| < 1$

In this region $f(z)$ is analytic

$$f(z) = \frac{1}{z(1-z)^2} = \frac{1}{z}(1-z)^{-2}$$
$$= \frac{1}{z}[1 + 2z + 3z^2 + \cdots]$$

This is Laurent series expansion is valid when $0 < |z| < 1$.

(ii) Consider the region $1 < |z| < \infty$

$f(z)$ is analytic in this region

$$f(z) = \frac{1}{z(1-z)^2} = \frac{1}{z^3\left(\frac{1}{z}-1\right)^2}$$
$$= +\frac{1}{z^3}\left(1-\frac{1}{z}\right)^2$$
$$= \frac{1}{z^3}\left(1+\frac{2}{z}+\frac{3}{z^2}+\cdots\right)$$

This is Laurent's series expansion valid when $|z| > 1$.

(iii) Consider the region $0 < |z-1| < 1$

$f(z)$ is analytic in this region.

$$f(z) = \frac{1}{(1-z)^2(1+z-1)}$$
$$= \frac{1}{(z-1)^2}[1+(z-1)]^{-1}$$
$$= \frac{1}{(z-1)^2}[1-(z-1)+(z-1)^2+\cdots]$$
$$= \frac{1}{(z-1)^2} - \frac{1}{z-1} + 1 - (z-1) + \cdots$$

This is the required Laurent's series valid when $0 < |z-1| < 1$.

(iv) Consider $1 < |z-1| < \infty$

where $f(z)$ is analytic

$$\therefore \quad f(z) = \frac{1}{(z-1)^2(1+(z-1))}$$
$$= \frac{1}{(z-1)^3\left(1+\frac{1}{z-1}\right)}$$
$$= \frac{1}{(z-1)^3}\left(1+\frac{1}{z-1}\right)^{-1}$$
$$= \frac{1}{(z-1)^3}\left(1-\frac{1}{z-1}+\frac{1}{(z-1)^2}+\cdots\right)$$

This is the Laurent's series expansion valid in the region $1 < |z-1| < \infty$.

Example 5.50

Find the Laurent's series expansion of the function $f(z) = \dfrac{1}{z(1+z^2)}$ in powers of z and specify the region in which the expansion is valid. **(AU 2008)**

Solution:

The function $f(z)$ is not analytic at $z = 0$ and $z = \pm i$. To expand by Laurent's series we have to find an annular region which does not include 0 and $\pm i$. We have to find the expansions in powers of z.

(i) Consider $0 < |z| < 1$, where $f(z)$ is analytic

$$\begin{aligned}\therefore \quad f(z) &= \frac{1}{z(1+z^2)} \\ &= \frac{1}{z}(1+z^2)^{-1} \\ &= \frac{1}{z}(1 - z^2 + z^4 - z^6 + \cdots) \\ &= \frac{1}{z} - z + z^3 - z^5 \cdots\end{aligned}$$

This is valid when $|z^2| < 1$ or $|z| < 1$.

(ii) Consider $1 < |z| < \infty$, where $f(z)$ is analytic

$$\begin{aligned}f(z) &= \frac{1}{z(1+z^2)} = \frac{1}{z^3\left(1+\frac{1}{z^2}\right)} \\ &= \frac{1}{z^3}\left(1+\frac{1}{z^2}\right)^{-1} \\ &= \frac{1}{z^3}\left(1 - \frac{1}{z^2} + \frac{1}{z^4}\cdots\right) \\ &= \frac{1}{z^3} - \frac{1}{z^5} + \frac{1}{z^7}\cdots\end{aligned}$$

This is valid when $\left|\dfrac{1}{z^2}\right| < 1$ or $|z| > 1$

Example 5.51

Find the Laurent's series expansion at $f(z) = \dfrac{3z-2}{(z^2-2z)z}$ in the annular region $2 < |z+2| < 4$. **(AU 2011)**

Solution:

$$f(z) = \frac{3z-2}{z(z^2-2z)}$$
$$= \frac{3z-2}{z^2(z-2)}$$

put $z+2=u$

$$\therefore \quad f(z) = \frac{3(u-2)-2}{(u-2)^2(u-2-2)}$$
$$= \frac{3u-8}{(u-2)^2(u-4)}$$
$$= \frac{-1}{u-2} + \frac{1}{(u-2)^2} + \frac{1}{u-4} \qquad \text{(by partial fractions)}$$
$$= \frac{-1}{u\left(1-\frac{2}{u}\right)} + \frac{1}{u^2\left(1-\frac{2}{u}\right)^2} - \frac{1}{4\left(1-\frac{u}{4}\right)}$$
$$= \frac{-1}{u}\left(1-\frac{2}{u}\right)^{-1} + \frac{1}{u^2}\left(1-\frac{2}{u}\right)^{-2} - \frac{1}{4}\left(1-\frac{u}{4}\right)^{-1}$$
$$= -\frac{1}{u}\left(1+\frac{2}{u}+\left(\frac{2}{u}\right)^2+\cdots\right) + \frac{1}{u^2}\left(1+2\left(\frac{2}{u}\right)+3\left(\frac{2}{u}\right)^2+\cdots\right)$$
$$-\frac{1}{4}\left(1+\left(\frac{u}{4}\right)+\left(\frac{u}{4}\right)^2+...\right)$$

The first and second expansion are valid when $\left|\frac{2}{u}\right| < 1$ or $2 < |u|$ and the third expansion is valid when $\left|\frac{u}{4}\right| < 4$ or $|u| < 4$. Hence in the annular region specified $2 < |u| < 4$ or $2 < |z+2| < 4$, the Laurents expansion is

$$f(z) = \frac{-1}{z+2}\left[1+\frac{2}{z+2}+\left(\frac{2}{z+2}\right)^2+\cdots\right] + \frac{+1}{(z+2)^2}\left[1+2\left(\frac{2}{z+2}\right)+3\left(\frac{2}{z+2}\right)^2+\cdots\right]$$
$$-\frac{1}{4}\left[1+\left(\frac{z+2}{4}\right)+\left(\frac{z+2}{4}\right)^2+...\right]$$

Exercise 5(b)

☞ **Part - A**

1. State Taylor's theorem.
2. State Laurent's theorem.

3. What is the analytic part and principal part of the Laurent's series of a function of z ?

4. Define convergence of a power series.

5. Find the Taylor series for the following functions.
   (i) $f(z) = \cos z$ about $z = \pi/3$
   (ii) $f(z) = e^z$ about $z = -i$
   (iii) $f(z) = e^{-z}$ about $z = 1$
   (iv) $f(z) = e^{2z}$ about $z = 2i$
   (v) $f(z) = \cos z$ about $z = -\pi/2$.

6. Without expanding, find the region of convergence in each of the following
   (i) $f(z) = \dfrac{z+3}{(z-1)(z-4)}$ about $z = 2$.
   (ii) $f(z) = \sec z$ about $z = 1$
   (iii) $f(z) = \dfrac{z-1}{z+1}$ about $z = 0$
   (iv) $f(z) = e^{2z}$ about $z = 2i$
   (v) $f(z) = \log\left(\dfrac{1+z}{1-z}\right)$ at $z = 0$.

7. Find the Laurent's series of each of the following functions.
   (i) $f(z) = \dfrac{1}{z(z-1)}$ valid in $0 < |z-1| < 1$.
   (ii) $f(z) = \dfrac{1}{z(1-z)}$ valid in $|z+1| < 1$.
   (iii) $f(z) = \dfrac{1}{z^3(1-z)}$, valid in $|z| > 1$.

☞ **Part - B**

8. Obtain the Taylor's series for the functions given below also state the region of convergence in each case.
   (i) $f(z) = \dfrac{2z^3+1}{z(z+1)}$ about the point $z = -i$
   (ii) $f(z) = \dfrac{z}{(z+1)(z+2)}$ about the point $z = 0$
   (iii) $f(z) = \dfrac{z^3+2z^2}{z^2+2z+3}$ in $|z+\frac{1}{2}| < 1$

9. Find the Laurent's expansion for function f(z) given below in the specified region.

   (i) $f(z) = \dfrac{4z+4}{z(z-3)(z+2)}$ in

   (a) $0 < |z| < 2$ (b) $2 < |z| < 3$ (c) $|z| > 3$

   (ii) $f(z) = \dfrac{z+3}{z(z^2-z+2)}$

   (a) $0 < |z| < 1$ (b) $0 < |z| < 2$ (c) $1 < |z| < 2$

   (iii) $f(z) = \dfrac{z^2-1}{z^2+5z+6}$

   (a) $|z| > 3$ (b) $|z| < 2$ (c) $z < |z| < 3$

   (iv) $f(z) = \dfrac{z}{(z-1)(z-2)}$ in

   (a) $|z+2| < 3$ (b) $3 < |z+2| < 4$ (c) $|z+2| > 4$

   (v) $f(z) = \dfrac{1}{(z+1)(z+3)}$ in

   (a) $1 < |z| < 3$ (b) $0 < |z+| < 2$

   (c) $|z| > 3$ (d) $|z| < 1$

   (vi) $f(z) = \dfrac{1}{(z^2+1)(z^2+2)}$ in

   (a) $|z| < 1$ (b) $1 < |z| < \sqrt{2}$ (c) $|z| > \sqrt{2}$

   (vii) $f(z) = \dfrac{z^2-6z-1}{(z-1)(z-3)(z+2)}$ in $3 < |z+2| < 5$

   (viii) $f(z) = \dfrac{e^{2z}}{(z-1)^3}$ in $|z-1| > 1$

   (ix) $f(z) = \dfrac{1}{(z-1)(z-2)}$ in

   (a) $|z-1| < 1$ (b) $|z| > 2$

   (x) $f(z) = \dfrac{z}{(z-1)(z-3)}$ in $0 < |z-1| < 2$

10. Expand $f(z) = \dfrac{z+3}{z(z^2-z-2)}$ in powers of z.

    (a) within the unit circle about the origin.

    (b) within the annular region between the concentric circle about the origin having radii 1 and 2 respectively.

(c) the exterior to the circle with centre as origin and radius 2.

11. Obtain the Laurent's series expansion for $\dfrac{1}{z(1-z)^2}$ about $z = 0$ and specify the regions in which the expansion are valid.

12. Represent $f(z) = \dfrac{z+1}{z-1}$ by

    (a) Its Laurent's series in powers of z for the region $|z| > 1$

    (b) Its Taylor's series in powers of z and give the region of validity.

    (c) Its Maclaurin's series and give the region for its validity.

5.5 Singularities

Definitions

Zeros of an analytic function: A *zero* of an analytic function $f(z)$ is a value of z for which $f(z) = 0$.

Let $f(z)$ be a function which is analytic in a region R and 'a' is any point in R then $f(z)$ is said to have a *zero of order m at z=a* if $f(z) = (z-a)^m\phi(z)$, where $\phi(z)$ is analytic at a and $\phi(a) \neq 0$.

■ **Note:** If $f(z)$ has a zero of order 1 at $z = a$ then $f(z)$ is said to have a simple zero at $z = a$.

For example

1. Let $f(z) = \sin z$

   By Taylor's series expansion about $z = 0$, we have

$$\begin{aligned}\sin z &= z - \frac{z^3}{3!} + \frac{z^5}{5!} + ... \\ &= z\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots\right) \\ &= z\phi(z)\end{aligned}$$

   $\phi(z)$ is analytic and $\phi(0) = 1 \neq 0$.

   $\therefore \quad z = 0$ is a zero of order 1 for $f(z)$ or $\sin z$ has a simple zero at $z = 0$.

2. Consider $f(z) = z\sin z$

By Taylor's expansion about $z = 0$, we have

$$f(z) = z\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right]$$
$$= z^2\left[1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right]$$
$$= z^2\phi(z)$$

$\phi(z)$ is analytic and $\phi(0) \neq 0$.

3.
$$f(z) = (z^2 - 1)(z^2 - 3z + 2)$$
$$= (z + 1)(z - 1)(z - 2)(z - 1)$$
$$= (z + 1)(z - 2)(z - 1)^2$$

$\therefore\ z = -1, 2$ are zero's of order 1 and $z = 1$ is a zero of order 2.

Singularities of an Analytic Function

A point 'a' is called a *singular point* of an analytic function $f(z)$, if $f(z)$ is not analytic at 'a' or $f(z)$ is said to have a *singularity* at 'a'.

Types of singularities

1. Isolated singularity:

A singular point $z = a$ of $f(z)$ is said to be an *isolated singular point* of $f(z)$ if there exists a circle $|z - a| = \delta, \delta > 0$ encloses no other singular point other than 'a'. In other words, the point $z = a$ is called an isolated singularity of $f(z)$, if there is no other singularity of $f(z)$, if there is no other singularity in its neighbourhood.

■ **Note:** If 'a' is not singular point and we can find $\delta > 0$ such that $|z - a| = \delta$ encloses no singular point, then we call 'a' an ordinary point of $f(z)$.

For example:

$f(z) = \dfrac{z+1}{(z-1)(z-2)}$ has two isolated singularities $z = 1$, $z = 2$ and $f(z) = \dfrac{z+3}{z^2(z^2+2)}$ than three isolated singularities $z = 0$, $z = \pm\sqrt{2}i$.

2. Pole

If a is an isolated singularity of $f(z)$. The point 'a' is called a pole if the principal part of $f(z)$ at $z = a$ valid in $0 < |z - a| < \delta$ has a finite number of terms.

If the principal part of $f(z)$ at $z = a$ is given by

$$\sum_{n=1}^{\infty} b_n(z-a)^{-n} = \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots + \frac{b_m}{(z-a)^m} + \cdots$$

where $b_m \neq 0$ then we say that 'a' is a **pole of order m**.

■ **Note:** A pole of order 1 is called a *simple pole* and a pole of order 2 is called a *double pole*.

For example

$f(z) = \dfrac{z^2 - 2z + 3}{z - 2}$ and $f(z) = \dfrac{e^z}{z}$ have simple poles at $z = 2$ and $z = 0$ respectively and $f(z) = \dfrac{3z}{(z-1)^2}$ has a pole at $z = 1$ of order 2.

3. Removable singularity

A singularity $z = a$ is called removable singularity of $f(z)$ if $\lim\limits_{z\to 0} f(z)$ exists.

For example

$$\begin{aligned} f(z) &= \frac{\sin(z-a)}{(z-a)} \\ &= \frac{1}{z-a}\left[(z-a) - \frac{(z-a)^3}{3!} + \frac{(z-a)^5}{5!} - \cdots\right] \\ &= 1 - \frac{(z-a)^2}{3!} + \frac{(z-a)^4}{5!} - \cdots \end{aligned}$$

$$\lim_{z\to a} \frac{\sin(z-a)}{(z-a)} = 1$$

$\therefore\ z = a$ is a removable singularity.

4. Essential singularity

If the principal part of $f(z)$ in its Laurent's series of $f(z)$ at $z = a$, valid in $0 < |z-a| < r$, has infinite number of terms, then $z = a$ is called *essential singularity*.

For example

1. $f(z) = e^{1/z} = 1 + \dfrac{1}{z} + \dfrac{1}{2!}\cdot\dfrac{1}{z^2} + \dfrac{1}{3!}\cdot\dfrac{1}{z^3} + ...\infty$

   hence $z = 0$ is an essential singularity.

2. $f(z) = \sin\left(\dfrac{1}{z-a}\right)$

   $$= \frac{1}{z-a} - \frac{1}{3!}\frac{1}{(z-a)^3} + \frac{1}{5!}\frac{1}{(z-a)^5} ...\infty$$

   hence $z = a$ is a essential singularity.

Entire function

A function $f(z)$ which is analytic everywhere in the finite plane (except at infinity) is called an *entire function* or an integral function.

For example $e^z, \sin z, \cos z$ are entire functions.

Meromorphic function

A function $f(z)$ which is analytic everywhere in the finite plane except at finite number of poles is called a *meromorphic function*.

For example $f(z) = \dfrac{z}{z(z-1)^2}$ is a *meromorphic function*, as it has only two poles a single pole at $z = 0$ and a double pole at $z = 1$.

5.6 Residues

If $z = a$ is an isolated singularity of $f(z)$, we can find the Laurent series of $f(z)$ about $z = a$.

i.e. $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} \dfrac{b_n}{(z-a)^n}$, then the coefficient of $\dfrac{1}{z-a}$ i.e. b_1 in the Laurent's series of $f(z)$ at $z = a$ valid in $0 < |z-a| < r$ is called of the *residue* of $f(z)$ at $z = a$.

Hence the residue of $f(z)$ at $z = a$ is also given by

$$[\text{Res} \quad f(z)]_{z=a} = \frac{1}{2\pi i} \int_C f(z)dz$$

where C is any closed curve around 'a' such that $f(z)$ is analytic within and on C, except at $z = a$.

1. If $z = a$ is a simple pole of $f(z)$, then

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \frac{b_1}{z-a}$$

$$(z-a)f(z) = \sum_{n=0}^{\infty} a_n(z-a)^{n+1} + b_1$$

$$\therefore \quad \lim_{z \to a} [(z-a)f(z)] = b_1 = [\text{Res} \quad f(z)]_{z=a}$$

Hence if $z = a$ is a simple pole of $f(z)$, then

$$\text{Residue of } f(z) \text{ at } z = a = \lim_{z \to a} (z-a)f(z)$$

2. If $z = a$ is a simple pole of $f(z)$ and if $f(z) = \dfrac{P(z)}{Q(z)}$ then

   $Q(z) = (z-a)\ R(z)$ where $R(a) \neq 0$ then $(z-a)$

   $f(z) = (z-a)\dfrac{P(z)}{Q(z)} = \dfrac{P(z)}{R(z)}$

$$\begin{aligned} \therefore \lim_{z \to a}(z-a)f(z) &= \lim_{z \to a}\left[\frac{(z-a)P(z)}{Q(z)}\right] \\ &= \lim_{z \to a}\left[\frac{(z-a)P'(z) + P(z)}{Q'(z)}\right] \\ &= \lim_{z \to a}\left[\frac{P(z)}{Q'(z)}\right] \end{aligned}$$

   Hence for a simple pole $z = a$ Residue of $f(z)$ at $z = a = \lim_{z \to a}\dfrac{P(z)}{Q'(z)}$.

3. If $z = a$ is a pole order m, then

$$(z-a)^m f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{m} b_n(z-a)^{-n}$$

$$\begin{aligned} \therefore (z-a)^m f(z) = \sum_{n=0}^{\infty} a_n(z-a)^{m+n} &+ b_1(z-a)^{m-1} \\ &+ b_2(z-a)^{m-2} + \cdots b_m \end{aligned}$$

   Differentiating $(m-1)$ times and taking the limit as $z \to a$, we get,

$$\text{Residue of } f(z) \text{ at } z = a = \lim_{z \to a}\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]$$

5.7 Cauchy's Residue Theorem

Statement: If C is a closed curve, and $f(z)$ is analytic within and on C, except at a finite number of singular points within C, then

$$\int_C f(z)dz = 2\pi i[R_1 + R_2 + \cdots + R_n]$$

where $R_1, R_2 \cdots R_n$ are the residues of $f(z)$ at the singular points $z_1, z_2 \cdots z_n$.

Proof:

$f(z)$ is an analytic function within and on C except for a finite number of singularities, $z, z_1 \cdots z_n$. We enclose the singularities by small non - intersecting circles $C_1, C_2, \cdots C_n$ with centre at $z_1, z_2, \cdots z_n$ and radii $r_1, r_2 \cdots r_n$ lying wholy within C.

Then $f(z)$ is analytic in the multiply connected region enclosed by the curves $C_1, C_2 \cdots C_n$. Hence by Cauchy's extension of integral theorem we have

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \cdots \int_{C_n} f(z)dz \quad (1)$$
$$= 2\pi i[R_1 + R_2 + \cdots + R_n] \quad \text{(by definition of Residue)}$$

Example 5.52

Give the formula of obtain the residue of $f(z)$ that has a pole of order m at $z = a$.
(AU 2009)

Solution:

Residue at a pole of order m is given by

$$[\text{Res } f(z)]_{z=a} = \underset{z\to a}{\text{Lt}} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z-a)^m f(z)]$$

Example 5.53

Find the nature and location of the singularity for function $f(z) = \dfrac{z - \sin z}{z^2}$.
(AU 2011)

Solution:

$$f(z) = \frac{z - \sin z}{z^2}$$

Hence $z = 0$ is the only singularity of $f(z)$

$$z - \sin z = z - \left(z - \frac{z^3}{\lfloor 3} + \frac{z^5}{\lfloor 5} \cdots\cdots \right)$$
$$= \frac{z^3}{\lfloor 3} + \frac{z^5}{\lfloor 5} + \cdots$$
$$\lim_{z\to 0} \left(\frac{z - \sin z}{z^2} \right) = 0$$

$\therefore \quad z = 0$ is a removable singularity

Example 5.54

Find the singularities of the following function and classify them in each case.

(i) $f(z) = \dfrac{z}{e^z - 1}$ (ii) $f(z) = \sin\left(\dfrac{1}{z}\right)$ (iii) $f(z) = \dfrac{\cot \pi z}{(z-a)^3}$

(iv) $f(z) = \dfrac{\sin z - z}{z^3}$ (v) $f(z) = \begin{cases} e^z & z \neq 0 \\ 0 & z = 0 \end{cases}$

(vi) $f(z) = \dfrac{z-2}{z^2} \sin\left(\dfrac{1}{z-1}\right)$

Solution:

(i) Given $f(z) = \dfrac{z}{e^z - 1}$

The singularies of $f(z)$ are given by the values of z for which $e^z - 1 = 0$

$\therefore\ e^z = 1 = e^{2n\pi i},\ (n = 0, \pm 1, \pm 2, \cdots)$

$\therefore\ z = 2n\pi i$ are the singularities of $f(z)$ now

$$e^z - 1 = \left[1 + z + \frac{z^2}{2!} + \cdots\right] - 1$$
$$= \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} - \cdots$$

$\therefore \lim\limits_{z \to 0} \left[\dfrac{z}{e^z - 1}\right] = +1$, hence $z = 0$ is a removable singularity for $f(z)$ also $\lim\limits_{z \to 2n\pi i} \left[\dfrac{z}{e^z - 1}\right] = \infty$ if $n \neq 0$, hence $z = 2n\pi i$ are simple poles of $f(z)$.

(ii) Given $f(z) = \sin\left(\dfrac{1}{z}\right)$

Here $z = 0$ is the only singularity of $f(z)$

$$\text{now} \qquad \sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} \cdots$$

$\lim\limits_{z \to 0} \sin\left(\dfrac{1}{z}\right)$ does not exist and the principal part of $f(z)$ contain an infinite number of terms.

Hence $z = 0$ is an essential singularity.

(iii) Given $f(z) = \dfrac{\cot \pi z}{(z-a)^3}$

$$\text{or} \qquad f(z) = \frac{\cot \pi z}{(z-a)^3 \sin \pi z}.$$

The singularities of $f(z)$ are given by the value of z for which $(z-a)^3 \sin \pi z = 0$.

Hence $z = a$ and $\sin \pi z = 0 \Rightarrow z = 0, \pm 1, \pm 2, \cdots$ are the singlularities of $f(z)$.

Hence $z = a$ is a pole of order 3 of $f(z)$ and $z = 0, \pm 1, \pm 2, \cdots$ are simple poles (each)

(iv) Given $f(z) = \dfrac{\sin z - z}{z^3}$

Here $z = 0$ is the only singularity of $f(z)$.

$$\therefore \quad \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = -\frac{z^3}{3!} + \frac{z^5}{5!} \cdots$$

$$\therefore \quad \sin z - z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

$$\therefore \quad \lim_{z \to 0} \left(\frac{\sin z - z}{z^3} \right) \neq 0$$

Hence $z = 0$ is a removable singularity of $f(z)$.

(v) Given $f(z) = \begin{cases} e^z & z \neq 0 \\ 0 & z = 0 \end{cases}$

given $f(z) = 0$ for $z = 0$

$\therefore \quad z = 0$ is the only singularity

$$\therefore \quad \lim_{z \to 0} f(z) = 1$$

Hence $z = 0$ is a removable singularity.

(vi) Given $f(z) = \dfrac{z-2}{z^2} \sin \left(\dfrac{1}{z-1} \right)$

The singularities are $z = 0$ and $z = 1 + \dfrac{1}{n\pi}, (n = 0, \pm 1, \pm 2, \cdots)$

Hence $z = 0$ is a pole of order 2 and $z = 1 + \dfrac{1}{n\pi}(n = 0, \pm 1, \pm 2, \cdots)$ is a essential singularity.

As the limit does not exists and the principal part contains in limits number of terms.

Example 5.55

Find the residue of $\dfrac{1 - e^{2z}}{z^4}$ at $z = 0$. **(AU 2013)**

Solution:

$$
\begin{aligned}
f(z) &= \frac{\left[-\left(+\frac{2z}{\underline{|1}}+\frac{(2z)^2}{\underline{|2}}+\frac{(2z)^3}{\underline{|3}}+.......\right)\right]}{z^4} \\
&= -\frac{2}{z^3}-\frac{2}{z^2}-\frac{8}{6z}+..... \\
[\text{Res} f(z)]_{z=0} &= \text{Coeff of} \frac{1}{z} = -\frac{8}{6} = -\frac{4}{3}
\end{aligned}
$$

Example 5.56

Expand $f(z) = \dfrac{1}{(z+1)(z+3)}$ in Laurent's series valid for the region $|z| > 3\ \&| < |z| < 3$. **(AU 2012)**

Solution:

$$
\begin{aligned}
\frac{1}{(z+1)(z+3)} &= \frac{A}{z+1}+\frac{B}{z+3} \\
A = 1/2,&\quad B = -1/2 \\
\therefore\ f(z) &= \frac{1/2}{z+1} - \frac{-1/2}{z+3}
\end{aligned}
$$

Case 1: $|z| > 3$

$$
\begin{aligned}
f(z) &= \frac{1/2}{z(1+1/z)} - \frac{1/2}{z(1+3/z)} \\
&= \frac{1}{2z}(1+1/z)^{-1} - \frac{1}{2z}(1+3/z)^{-1} \\
&= \frac{1}{2z}\sum(-1)^n\left(\frac{1}{z}\right)^n - \frac{1}{2z}\sum(-1)^n(3/z)^n
\end{aligned}
$$

This is valid $\left|\frac{1}{z}\right| < |\quad \& \quad |3/z| < 1$

i.e., $|z| > 1$ i.e., $|z| > 3$

i.e., $|z| > 3$

Case 2: $1 < |z| < 3$

i.e., $|z| > 1$ and $|z| < 3$

$$
\begin{aligned}
f(z) &= \frac{1/2}{z+1} - \frac{1/2}{z+3} \\
&= \frac{1/2}{z(1+1/z)} - \frac{1/2}{3(1+z/3)} \\
&= \frac{1}{2z}(1+1/2)^{-1} - \frac{1}{6}(1+2/3)^{-1}
\end{aligned}
$$

$$= \frac{1}{2z}\sum(-1)^n\left(\frac{1}{z}\right)^n - \frac{1}{6}\sum(-1)^n(2/3)^n$$

This is valid $\left|\frac{1}{z}\right| < 1 \quad \& \quad |z/3| < 1$

i.e., $|z| > 1 \qquad |z| < 3$

$\therefore \quad 1 < |z| < 3$

Example 5.57

Find the residues of the function $f(z) = \dfrac{4}{z^3(z-2)}$ at a simple pole. **(AU 2012)**

Solution:

$$z = 2 \text{ is a simple pole}$$

$$[\text{Res } f(z)]_{z=2} = \lim_{z\to 2}(z-2)f(2)$$

$$= \lim_{z\to 2}(z-2)\frac{4}{z^3(z-2)}$$

$$= \frac{4}{z^3} = \frac{4}{2^3} = \frac{4}{8} = 1/2$$

Example 5.58

Determine the residue of $f(z) = \dfrac{z^2}{(z-1)^2(z+2)}$ at $z = 1$. **(AU 2011)**

Solution:

$$z = 1 \text{ is a pole of order } 2$$

$$\therefore \quad (\text{Res } f(z) = \lim_{z\to 1}\frac{d}{dz}\left[(z-1)^2\frac{z^2}{(z+2)(z-1)^2}\right]$$

$$= \lim_{z\to 1}\frac{(z+2)2z - z^2(1)}{(z+2)^2}$$

$$= \frac{6-1}{9} = 5/9$$

Example 5.59

Find the residues of $f(z) = \dfrac{z^2}{(z-1)(z+2)^2}$ at its singular points. **(AU 2011)**

Solution:

$$\text{Put } (z-1)(z+2)^2 = 0$$

$$\therefore \quad z = 1 \text{ is a pole of order } 1$$

$$z = -2 \text{ is a pole of order } 2$$

$$(\text{Res } f(z))_{z=1} = \lim_{z\to 1} (z-1)\frac{z^2}{(z-1)(z+2)^2} = \frac{1}{9}$$

$$(\text{Res } f(z))_{z=-2} = \lim_{z\to -2} \frac{d}{dz}\left\{(z+2)^2\frac{z^2}{(z-1)(z+2)^2}\right\}$$

$$= \lim_{z\to -2} \left\{\frac{(z-1)2z - z^2(1)}{(z-1)^2}\right\} = \frac{(-3)(-4)-4}{9}$$

$$= 8/9$$

Example 5.60

Find the residue of $\cot z$ at the pole $z = 0$. **(AU 2010)**

Solution:

$$\because \quad [\text{Res}\,(f(z))]_{z=a} = \lim_{z\to a}(z-a)\ f(z)$$

$$\text{Here} \quad f(z) = \cot z$$

$$= \frac{\cos z}{\sin z} = \frac{\phi(z)}{\psi(z)}$$

$$\& \quad z = a = 0$$

$$\phi(z) = \cos z$$

$$\phi(0) = \cos 0 = 1 \neq 0$$

$$\psi(z) = \sin z$$

$$\psi(0) = \sin 0 = 0$$

$$\therefore \quad [\text{Res}\,(\cot z)]_{z=0} = \lim_{z\to 0}(z-0)\ f(z)$$

$$= \frac{\phi(z)}{\psi'(z)} = \frac{\cos z}{\frac{d}{dz}(\sin z)} = \frac{\cos z}{\cos z} = 1$$

$$\therefore \quad [\text{Res}\,(\cot z)]_{z=0} = 1$$

Example 5.61

If $f(z) = \dfrac{-1}{z-1} - 2\left[1 + (z-1) + (z-1)^2 + \ldots\right]$, find the residue of $f(z)$ at $z = 1$. **(AU 2010)**

Solution:

$$[\text{Res}\ \ f(z)]_{z=1} = -1$$

Example 5.62

Calculate the residues of $f(z)$ at its poles where

(i) $f(z) = \dfrac{z^2}{(z-1)^2(z+2)}$ **(AU 2009)** (ii) $f(z) = \dfrac{z^2}{z^4+a^4}$

(iii) $f(z) = \dfrac{e^z}{z^2(z^2+9)}$ **(AU 2006)** (iv) $f(z) = \dfrac{z^2+4}{z^3+2z^2+2z}$

(v) $f(z) = \dfrac{1}{(z^2+a^2)^2}$ **(AU 2007)** (vii) $f(z) = \dfrac{ze^z}{(z-1)^3}$ **(AU 2009)**

Solution:

(i) Given $f(z) = \dfrac{z^2}{(z-1)^2(z+2)}$

Here $z = 1$ is a pole of order 2

and $z = -2$ is a pole of order 1

$$\begin{aligned}
\therefore \text{Res}\,[f(z)]_{z=-2} &= \lim_{z\to -2}(z+2)f(z) \\
&= \lim_{z\to -2}(z+2)\frac{z^2}{(z-1)^2(z+2)} \\
&= \frac{4}{9} \\
\text{and} \quad \text{Res}[f(z)]_{z=1} &= \frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}[(z-1)^2 f(z)] \\
&= \lim_{z\to 1}\frac{d}{dz}\left[(z-1)^2\frac{z^2}{(z-1)^2(z+2)}\right] \\
&= \lim_{z\to 1}\frac{d}{dz}\left[\frac{z^2}{z+2}\right] \\
&= \lim_{z\to 1}\left[\frac{(z+2)(2z)-(z^2)(1)}{(z+2)^2}\right] \\
&= \frac{5}{9}
\end{aligned}$$

(ii) Given $f(z) = \dfrac{z^2}{z^4+a^4}$

The poles of $f(z)$ are given by

$$z^4 + a^4 = 0$$

$$\text{or} \quad z = a(-1)^{1/4}$$
$$= a\cos\left[\frac{(2n+1)\pi}{4} + i\sin\frac{(2n+1)\pi}{4}\right]$$
$$= ae^{i(2n+1)\pi/4} \text{ where } n = 0, 1, 2, 3.$$

$\therefore \quad z = ae^{i\pi/4}, ae^{i3\pi/4}, ae^{i5\pi/4}, ae^{i7\pi/4}$ are all simples poles.

As $f(z)$ is of the form $\dfrac{P(z)}{Q(z)}$

$$[\text{Res } f(z)]_{z=ae^{i\pi/4}} = \lim_{z\to ae^{i\pi/4}}\left[\frac{P(z)}{Q'(z)}\right]$$
$$= \lim_{z\to ae^{i\pi/4}}\left[\frac{1}{4z}\right]$$
$$= \frac{1}{4a}e^{-i\pi/4}$$
$$= \frac{(1-i)}{4\sqrt{2}a}$$
$$[\text{Res} f(z)]_{\left(z=ae^{i3\pi/4}\right)} = \lim_{z\to ae^{i3\pi/4}}\left[\frac{1}{4z}\right] = \frac{1}{4a}e^{-i3\pi/4}$$
$$= \frac{-(1+i)}{4\sqrt{2}a}$$
$$[\text{Res} f(z)]_{\left(z=ae^{i5\pi/4}\right)} = \lim_{z\to ae^{i5\pi/4}}\left[\frac{1}{4z}\right]$$
$$= \frac{1}{4a}e^{-i5\pi/4} = \frac{1}{4\sqrt{2}a}(-1+i)$$
$$[\text{Res} f(z)]_{\left(z=ae^{i7\pi/4}\right)} = \lim_{z\to ae^{i7\pi/4}}\left[\frac{1}{4z}\right]$$
$$= \frac{1}{4a}e^{-i7\pi/4}$$
$$= \frac{1}{4\sqrt{2}a}(1+i).$$

(iii) Given $f(z) = \dfrac{e^z}{z^2(z^2+9)}$

Here $z = 0$ is a double pole of $f(z)$

and $z = 3i$ are simple poles of $f(z)$.

$$[\text{Res } f(z)]_{z=0} = \frac{1}{1!}\lim_{z\to 0}\frac{d}{dz}(z-0)^2 f(z)$$

$$= \lim_{z\to 0} \frac{d}{dz}\left(\frac{e^z}{z^2+9}\right)$$

$$= \lim_{z\to 0} e^z\left[\frac{(z^2+9)-2z}{(z^2+9)^2}\right]$$

$$= \frac{1}{9}$$

$$[\text{Res} f(z)]_{z=3i} = \lim_{z\to 3i}(z-3i)f(z)$$

$$= \lim_{z\to 3i}\frac{e^z}{(z^2)(z+3i)}$$

$$= \frac{e^{3i}}{-54i}$$

$$= \frac{ie^{3i}}{54}$$

Similarly $[\text{Res } f(z)]_{z=-3i} = \dfrac{ie^{-3i}}{54}$

(iv) Given $f(z) = \dfrac{z^2+4}{z^3+2z^2+2z}$

$$= \frac{z^2+4}{z(z^2+2z+2)} = \frac{z^2+4}{z(z+1-i)(z+1+i)}$$

Here $z = 0$ and $z = -1 \pm i$ are simple poles of $f(z)$ and $f(z)$ is of the for $\dfrac{P(z)}{Q(z)}$.

$$\therefore \quad [\text{Res} f(z)]_{z=0} = \lim_{z\to 0}\frac{P(z)}{Q'(z)}$$

$$= \lim_{z\to 0}\frac{z^2+4}{3z^2+4z+2}$$

$$= \frac{4}{2} = 2$$

$$[\text{Res} f(z)]_{z=-1+i} = \lim_{z\to -1+i}\left[\frac{P(z)}{Q'(z)}\right]$$

$$= \lim_{z\to -1+i}\left[\frac{z^2+4}{3z^2+4z+2}\right]$$

$$= \frac{-1+3i}{2}$$

Similarly $[\text{Res} f(z)]_{z=-1-i} = \dfrac{-(1+3i)}{2}$

(v) Given $f(z) = \dfrac{1}{(z^2+a^2)^2}$

Here $z = \pm ai$ are poles of order 2 for $f(z)$

$$\therefore \quad [\text{Res} f(z)]_{z=ai} = \frac{1}{1!} \lim_{z \to ai} \frac{d}{dz} \left[(z-ai)^2 \cdot f(z) \right]$$
$$= \frac{1}{1!} \lim_{z \to ai} \frac{d}{dz} \frac{1}{(z+ai)^2}$$
$$= \lim_{z \to ai} \frac{-2}{(z+ai)^3}$$
$$= \frac{-i}{4a^3}$$

$$\text{Similarly} \quad [\text{Res } f(z)]_{z=-ai} = \frac{+i}{4a^3}$$

(vi) Given $f(z) = \dfrac{ze^z}{(z-1)^3}$

Here $z = 1$ is a pole of order 3 for $f(z)$.

$$\therefore \quad [\text{Res} f(z)]_{z=1} = \lim_{z \to 1} \frac{d^2}{dz^2} [(z-1)^3 \cdot f(z)]$$
$$= \frac{1}{2} \lim_{z \to 1} \frac{d^2}{dz^2} [ze^z]$$
$$= \frac{1}{2} \lim_{z \to 1} e^z (z+2)$$
$$= \frac{3e^z}{2}$$

Example 5.63

Find the residue of $f(z) = \dfrac{50z}{(z+4)(z-1)^2}$ at $z = 1$. **(AU 2009)**

Solution:

$$f(z) = \frac{50z}{(z+4)(z-1)^2}$$

$z = 1$ is a pole of order 2.

The residue of $f(z)$ at $z = 1$ is

$$= \lim_{z \to 1} \frac{d}{dz} (z-1)^2 \frac{50z}{(z+4)(z-1)^2}$$
$$= \lim_{z \to 1} \frac{(z+4)50 - 50z(1)}{(z+4)^2}$$

$$= \frac{5(50) - 50(1)}{25}$$
$$= \frac{200}{25} = 8$$

Example 5.64

Evaluate the following, using Cauchy's residue theorem

(i) $\int_C \frac{4-3z}{z(z-1)(z-2)} dz$, where C is $|z| = 3/2$ **(AU 2009)**

(ii) $\int_C \frac{z-2}{z(z-1)} dz$, where C is $|z| = 2$

(iii) $\int_C \frac{dz}{z \sin z}$, where C is $|z| = 1$ **(AU 2009)**

(iv) $\int_C \frac{e^z dz}{(z^2+\pi^2)^2}$, where C is $|z| = 4$ **(AU 2010)**

(v) $\int_C \tan z dz$, where C is $|z| = 2$ **(AU 2011)**

Solution:

(i) Let $f(z) = \frac{4-3z}{z(z-1)(z-2)}$

Here the poles of $f(z)$ are $z = 0, 1, 2$ which are simple poles.

Given $C : |z| = 3/2$ which represents a circle with centre at 0 and radius 3/2. Hence out of the three poles only $z = 0$ $z = 1$ lies within C.

$$\therefore \text{ Let } \quad R_1 = [\text{Res } f(z)_{z=0}] = \lim_{z\to 0}(z-0)f(z)$$
$$= \lim_{z\to 0} \frac{4-3z}{(z-1)(z-2)}$$
$$= \frac{4}{(-1)(-2)} = 2$$
$$\text{and } \quad R_2 = [\text{Res } f(z)]_{z=1} = \lim_{z\to 1}(z-1)f(z)$$
$$= \lim_{z\to 1} \frac{4-3z}{(z)(z-2)} = -1$$

$\therefore$ By Cauchy's residue theorem

$$\begin{aligned}\int_C f(z)dz &= 2\pi i[R_1 + R_2]\\ &= 2\pi i(2-1)\\ &= 2\pi i\end{aligned}$$

$$\int_C \frac{4-3z}{z(z-1)(z-2)}dz = 2\pi i$$

(ii) $\int_C \frac{z-2}{z(z-1)}dz C : |z| = 2$

Let $f(z) = \frac{z-2}{z(z-1)}$

Here, the poles of $f(z)$ are $z = 0$ and $z = 1$ which are simple poles
Hence $z = 0$ and $z = 1$ line within $|z| = 2$

$$\text{Let} \quad R_1 = [\text{Res} f(z)]_{z=0} = \lim_{z\to 0}(z-0)\cdot f(z) = \lim_{z\to 0}\frac{(z-2)}{z-1} = 2$$

$$\text{and} \quad R_2 = [\text{Res} f(z)]_{z=1} = \lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}\frac{z-2}{z} = -1$$

By Cauchy's residue theorem

$$\begin{aligned}\int_C f(z)dz &= 2\pi i[R_1 + R_2]\\ &= 2\pi i(2-1)\\ &= 2\pi i\end{aligned}$$

$$\therefore \quad \int_C \frac{z-2}{z(z-1)}dz = 2\pi i$$

(iii) Given $\int_C \frac{dz}{z\sin z}, C : |z| = 1$

Let $f(z) = \frac{1}{z\sin z}$

The poles of $f(z)$ are gives by $z\sin z = 0$

$$\text{i.e.}\qquad z\left[z - \frac{z^3}{3!} + \frac{z^5}{5!}\cdots\right] = 0$$

$$\text{or}\qquad z^2\left[1 - \frac{z^2}{3!} + \frac{z^4}{5!}\cdots\right] = 0$$

$\therefore$ $z = 0$ is a double pole of (z), which lies with in $|z| = 1$.

$$\therefore\ [\text{Res} f(z)]_{z=0} = \frac{1}{1!,}\lim_{z\to 0}\frac{d}{dz}(z-0)^2\cdot f(z)$$

$$= \lim_{z\to 0}\frac{d}{dz}\left[\frac{2}{\sin z}\right]\left(\frac{0}{0}\text{form}\right)\quad \text{(using L'Hospital's rule)}$$

$$= \lim_{z\to 0}\left[\frac{\sin z - z\cos z}{\sin^2 z}\right]$$

$$= \lim_{z\to 0}\left[\frac{z}{2\cos z}\right] = 0$$

By Cauchy's residue theorem

$$\therefore\quad \int_C f(z)dz = 2\pi i[\text{Res}\quad f(z)]_{z=0} = 0$$

(iv) Given $\displaystyle\int_C \frac{e^z}{(z^2+\pi^2)^2} C:\ |\,z|\ =\ 4$

Let $f(z) = \displaystyle\int_C \frac{e^z}{(z^2+\pi^2)^2}$

Here the pole of $f(z)$ are $z = \pm\pi i$

Which are poles of order 2 and lie within C

$$\therefore\quad \text{Let}\quad R_1 = [\text{Res} f(z)]_{z=\pi i} = \frac{1}{1!}\lim_{z\to\pi i}\frac{d}{dz}\left[(z-\pi i)^2 f(z)\right]$$

$$= \lim_{z\to\pi i}\frac{d}{dz}\left[\frac{e^z}{(z+\pi i)^2}\right]$$

$$= \lim_{z\to\pi i}\left[\frac{(z+\pi i)^2 e^z - 2(z+\pi i)e^z}{(z+\pi i)^4}\right]$$

$$= \frac{e^{i\pi}(2\pi i - 2)}{8\pi i^4} = \frac{1}{4\pi^3}(\pi + i)$$

Similarly $\quad R_2 = [\text{Res} f(z)]_{z=-\pi i} = \dfrac{1}{4\pi^3}(\pi - i)$

By Cauchy's residue theorem

$$\int_C f(z)dz = 2\pi i(R_1 + R_2)$$

$$= 2\pi i\frac{[\pi + i + \pi - i]}{4\pi^3} = \frac{2\pi i}{4\pi^3}\cdot 2\pi = \frac{i}{\pi}$$

(v) Given $\int_C \tan z dz \;\; c : |z| = 2$

Let $f(z) = \tan z = \dfrac{\sin z}{\cos z}$

The poles of $f(z)$ are given by $\cos z = 0$.

i.e. $z = (2n+1)\dfrac{\pi}{2}$, where $n = 0, \pm 1, \pm 2, \cdots$ which are simple poles.

Out of these poles only $z = +\dfrac{\pi}{2}$ and $z = -\dfrac{\pi}{2}$ lie within C and $f(z)$ is of the form $\dfrac{P(z)}{Q(z)}$.

$$\therefore \quad \text{let } R_1 = [\text{Res} f(z)]_{z=\pi/2} = \lim_{z\to\frac{\pi}{2}} \frac{P(z)}{Q'(z)}$$

$$= \lim_{z\to\frac{\pi}{2}} \frac{\sin z}{-\sin z}$$

$$= -1$$

$$\text{Similarly} \quad R_2 = [\text{Res}\, f(z)]_{z=-\frac{\pi}{2}} = -1$$

$\therefore$ By Cauchy's residue theorem

$$\int_C f(z)dz = 2\pi i[R_1 + R_2]$$

$$= 2\pi i[-1-1]$$

$$= -4\pi i$$

$$\therefore \quad \int_C \tan z\, dz = -4\pi i$$

Example 5.65

Evaluate $\displaystyle\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$, where C is $|z| = 3$. **(AU 2013)**

Solution:

$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} = \int_C \left(\frac{1}{z-2} - \frac{1}{z-1}\right)(\cos \pi z^2 + \sin \pi z^2)dz$$

$$= \int_C \frac{(\cos \pi z^2 + \sin \pi z^2)}{z-2}dz - \int_C \frac{\cos \pi z^2 + \sin \pi z^2}{z-1}dz$$

$z = 2$ lies inside $|z| = 3$; $z = 1$ lies inside $|z| = 3$.

$$I = 2\pi i f(2) - 2\pi i f(1) \qquad \text{where } f(z) = \cos \pi z^2 + \sin \pi z^2$$

$$= 2\pi i[\cos 4\pi + i \sin 4\pi] - 2\pi i[\cos \pi + \sin \pi] \qquad \begin{matrix} \sin \pi = 0 \\ \cos \pi = (-1)^n \end{matrix}$$

$$= 2\pi i[1 + 0] - 2\pi i[-1 + 0]$$

$$= 2\pi i + 2\pi i = 4\pi i$$

Example 5.66

Evaluate the following using Cauchy's residue theorem

(i) $\displaystyle\int_C \frac{dz}{(z^2+4)^2}$ where C is $|z - i| = 2$

(ii) $\displaystyle\int_C \frac{12z-7}{(2z+z)(z-1)^2}dz$, where C is $|z + i| = \sqrt{3}$ **(AU 2006)**

(iii) $\displaystyle\int_C \frac{z}{\cos z}dz$, where C is $|z - \pi| = \frac{\pi}{2}$

(iv) $\displaystyle\int_C \frac{(z-1)}{(z+1)^2(z-2)}dz$, where C is $|z - i| = 2$ **(AU 2008, 2012)**

(v) $\displaystyle\int_C \frac{(z+1)}{z^2+2z+4}dz$, where C is $|z + 1 + i| = 2$ **(AU 2010)**

(vi) $\displaystyle\int_C \frac{e^t dz}{(z+2)(z-1)}$, where C is $|z - 1| = 1$

Solution:

(i) Given $\int_C \frac{dz}{(z^2+4)^2} C : |z-i| = 2$

Let $f(z) = \frac{1}{(z^2+4)^2} = \frac{1}{(z+2i)^2(z-2i)^2}$

The poles of $f(z)$ are $z = +2i$ and $z = -2i$, which are both poles of order 2.

Out of these two poles only $z = 2i$ lies within C

$$\text{Let} \quad R_1 = [\text{Res} f(z)]_{z=2i} = \frac{1}{1!} \lim_{z\to 2i} \frac{d}{dz}\left[(z-2i)^2 \cdot f(z)\right]$$
$$= \lim_{z\to 2i} \frac{d}{dz} \frac{1}{(z+2i)^2}$$
$$= \lim_{z\to 2i} \left[\frac{-2}{(z+2i)^3}\right] = \frac{1}{32i}$$

$\therefore$ By Cauchy's residue theorem

$$\int_C f(z)dz = 2\pi i(R_1) = 2\pi i\left(\frac{1}{32i}\right)$$
$$= \frac{\pi}{16}$$

(ii) Given $\int_C \frac{12z-7}{(2z+3)(z-1)^2} dz \;\; C : |z+i| = \sqrt{3}$

Let $f(z) = \frac{12z-7}{(2z+3)(z-1)^2}$

The poles of $f(z)$ are $z = 1$ and $z = -3/2$

where $z = 1$ is a pole of order 2

and $z - 3/2$ is pole of order 1

Out of these two poles only $z = 1$ lies within C.

$$\therefore \quad [\text{Res} f(z)]_{z=1} = \frac{1}{1!} \lim_{z\to 1} \frac{d}{dz}\left[(z-1)^2 \cdot f(z)\right]$$
$$= \lim_{z\to 1} \frac{d}{dz}\left[\frac{12z-7}{2z+3}\right]$$
$$= \lim_{z\to 1}\left[\frac{(2z+3)(12)-(12z-7)(2)}{(2z+3)^2}\right]$$
$$= \lim_{z\to 1}\left[\frac{50}{(2z+3)^2}\right] = 2$$

$\therefore$ By Cauchy's residue theorem

$$\int_C f(z)dz = 2\pi i(2) = 4\pi i$$

$$\therefore \quad \int_C \frac{12z-7}{(2z+3)(z-1)^2}dz = 4\pi i$$

(iii) Given $\displaystyle\int_C \frac{z}{\cos z}dz$ when $C : |z-\pi/2| = \pi/2$

Let $f(z) = \dfrac{z}{\cos z}$

The poles of $f(z)$ are given by $\cos z = 0$

i.e. $z = (2n+1)\pi/2, n = 0, \pm 1, \pm 2, \cdots$ which are simple poles

Out of these poles only $z = \pi/2$ line within C and $f(z)$ is of the form $\dfrac{P(z)}{Q(z)}$

$$\therefore \quad [\text{Res}(f(z))]_{z=\pi/2} = \lim_{z\to\pi/2}\left(\frac{P(z)}{Q(z)}\right) = \lim_{z\to\pi/2}\left(-\frac{z}{\sin z}\right) = -\pi/2$$

$\therefore$ By Cauchy's residue theorem

$$\int_C f(z)dz = 2\pi i\left(\frac{-\pi}{2}\right) = -\pi^2 i$$

(iv) $\displaystyle\int_C \frac{z+1}{z^2+2z+4}dz, c; |z+1+i| = 2$

Let $f(z) = \dfrac{z+1}{z^2+2z+4}$

the poles of $f(z)$ are given by $z^2+2z+4 = 0$

or $(z+1)^2+3 = 0$

$\therefore \quad z = -1 \pm i\sqrt{3}$ are the simple poles of $f(z)$. Out of these two poles only $z = -1 - i\sqrt{3}$ lies inside C.

$$\therefore [\text{Res}f(z)]_{z=-1-\sqrt{3}} = \lim_{z\to-(1+\sqrt{3i})}(z+1+i\sqrt{3})f(z)\left(\frac{z+1}{z+1-i\sqrt{3}}\right)$$

$$= \lim_{z\to-(1+\sqrt{3i})} = \frac{1}{2}$$

$\therefore$ By Cauchy's residue theorem

$$\int_C f(z)dz = 2\pi i(1/2) = \pi i$$

(v) $\displaystyle\int_C \frac{e^z}{(z+2)(z+1)}dz; C : |z-1| = 1$

Let $f(z) = \dfrac{e^z}{(z+2)(z+1)}$

$\therefore$ the poles of $f(z)$ are $z = -2$ and $z = 1$. $z = -2$ and $z = 1$ are simple poles of $f(z)$. Out of these two poles only $z = 1$ lies inside C.

$$\therefore \quad [\text{Res} f(z)]_{z=1} = \lim_{z\to 1}(z-1)\cdot f(z)$$
$$= \lim_{z\to 1}\left[\frac{e^z}{z+2}\right] = \frac{e^1}{3}$$

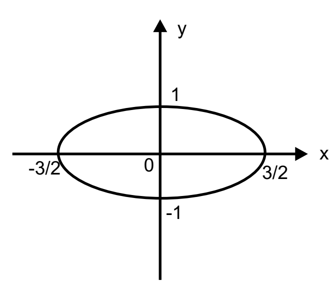
$\therefore$ By Cauchy's residence theorem

$$\int_C f(z)dz = 2\pi i\left(\frac{e}{3}\right) = \frac{i2\pi e}{3}$$

(vi) Evaluate $\displaystyle\int_C \frac{z\sec z}{(1-z^2)}dz$, where C is the ellipse $4x^2 + ay^2 = 9$, using Cauchy's residue theorem.

Solution:

Let $f(z) = \dfrac{z\sec z}{1-z^2} = \dfrac{z}{(1-z^2)\cos z}$



The poles of $f(z)$ are $z = \pm 1, z = (2n+1)\pi/2$.

$z = \pm 1$ and $z = (2n+1)\pi/2$, $n = 0, \pm 1, \pm 2\cdots$ are simple poles.

The ellipse $4x^2 + ay^2 = 9$, meets the real axis at $x = \pm 3/2$ and the imaginary axis at $y = \pm 1$.

Hence out of these poles only $z = \pm 1$ lie within C.

$$\text{Let} \quad R_1 = [\text{Res } f(z)]_{z=1}$$

$$= \lim_{z \to 1} (z-1) f(z)$$
$$= \lim_{z \to 1} \frac{-z}{(\cos z)(1+z)}$$
$$= \frac{-1}{2\cos 1} = \frac{-\sec 1}{2}$$

Similarly, $R_2 = [\text{Res} f(z)]_{z=-1} = \dfrac{-1}{2\cos 1} = \dfrac{-\sec 1}{2}$

$\therefore$ By Cauchy's residue theorem.

$$\int_C f(z) dz = 2\pi i [R_1 + R_2]$$
$$= 2\pi i \left[\frac{-\sec 1}{2} - \frac{\sec 1}{2} \right] = -2\pi i \sec 1.$$

Example 5.67

Evaluate $\displaystyle\int_C \frac{z-3}{z^2-z} dz$ where C is a circle $x^2+y^2=9$. Using Cauchy's residue theorem.

Solution:

Let $f(z) = \dfrac{z-2}{z^2-z} = \dfrac{z-2}{z(z-1)}$

Given C is a circle $x^2+y^2=9$ or $|z|=3$ the poles of $f(z)$ are $z=0$ and $z=1$, which are both simple poles. Both these poles lie with in C.

$$\text{Let} \quad R_1 = [\text{Res} f(z)]_{z=0} = \lim_{z \to 0} (z-0) f(z)$$
$$= \lim_{z \to 0} \left[\frac{z-2}{z-1} \right] = 2$$
$$\text{and} \quad R_2 = [\text{Res} f(z)]_{z=1} = \lim_{z \to 1} (z-1) \cdot f(z)$$
$$= \lim_{z \to 1} \left[\frac{z-2}{z} \right] = -1$$

$\therefore$ By Cauchy's residue theorem

$$\int_C f(z) dz = 2\pi i [R_1 + R_2]$$
$$= 2\pi i (2-1) = 2\pi i$$

Example 5.68

Use Residue theorem, to evaluate $\int_C \frac{3z^2+z-1}{(z^2-1)(z-3)}\,dz$ where C is the circle $|z|=4$. **(AU 2010)**

Solution:

$$\text{Here} \quad f(z) = \frac{3z^2+z-1}{(z^2-1)(z-3)} = \frac{3z^2+z-1}{(z-1)(z+1)(z-3)}$$

Here $z=-1,\ 1,\ 3$ is a pole of order 1

$z = 1,\ 3,\ -1$ lies inside the circle $|z|=2$

Now

$$\text{Res}\,[f(z)]_{z=1} = \underset{z\to 1}{Lt}\ (z-1)\ f(z)$$

$$= \underset{z\to 1}{Lt}\ (z-1)\cdot\frac{3z^2+z-1}{(z-1)(z+1)(z-3)}$$

$$= \frac{3}{2(-2)} = \frac{3}{-4}$$

$$\text{Res}\,[f(z)]_{z=-1} = \underset{z\to -1}{Lt}\ (z+1)\ f(z)$$

$$= \underset{z\to -1}{Lt}\ (z+1)\cdot\frac{3z^2+z-1}{(z-1)(z+1)(z-3)}$$

$$= \frac{3-1-1}{(-4)(-2)} = \frac{1}{8}$$

$$\text{Res}\,[f(z)]_{z=3} = \underset{z\to 3}{Lt}\ (z-3)\ f(z)$$

$$= \underset{z\to 3}{Lt}\ (z-3)\cdot\frac{3z^2+z-1}{(z-1)(z+1)(z-3)}$$

$$= \frac{3(9)+3-1}{(2)(4)} = \frac{29}{8}$$

$\therefore$ By Cauchy's Residue Theorem,

$$\int_C \frac{3z^2+z-1}{(z^2-1)(z-3)}\,dz = 2\pi i\ [\text{sum of the residues}]$$

$$= 2\pi i\left[\frac{-3}{4}+\frac{1}{8}+\frac{29}{8}\right]$$

$$= 2\pi i\left[\frac{-6+1+29}{8}\right]$$

$$= 6\pi i$$

Example 5.69

Evaluate $\int_C \frac{1}{z^3(z+4)} dz$ where C is the circle

(i) $|z| = 2$ (ii) $|z+2| = 3$. **(AU 2010)**

Solution:

Let $f(z) = \frac{1}{z^3(z+4)}$

The poles of $f(z)$ are $z = 0$ and $z = -4$. $z = 0$ is a pole of order 3 and $z = -4$ is a pole of order 1.

(i) Consider $C : |z| = 2$

Out of these two poles only $z = 0$ lies within C.

$$\begin{aligned}
\therefore \quad [\text{Res} f(z)]_{z=0} &= \frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2} \left[(z-0)^3 f(z)\right] \\
&= \frac{1}{2} \lim_{z \to 0} \frac{d^2}{dz^2} \left[\frac{1}{z+4}\right] \\
&= \frac{1}{2} \lim_{z \to 0} \left[\frac{2}{(z+4)^3}\right] = \frac{1}{64}
\end{aligned}$$

$\therefore$ By Cauchy's residue theorem

$$\int_C f(z) dz = 2\pi i \left(\frac{1}{64}\right) = \frac{\pi i}{32}$$

(ii) Consider $C : |z+2| = 3$

Both the poles $z = 0$ and $z = -4$ lie within c

$$\begin{aligned}
\therefore \quad R_1 &= [\text{Res} f(z)]_{z=0} = \frac{1}{64} \\
\text{and} \quad R_2 &= [Res f(z)]_{z=-4} = \lim_{z \to -4} (z+4) \cdot f(z) \\
&= \lim_{z \to -4} \left[\frac{1}{z^3}\right] \\
&= -\frac{1}{64}
\end{aligned}$$

$\therefore$ By Cauchy's residue theorem

$$\int_C f(z)dz = 2\pi i\,[R_1 + R_2]$$

$$= 2\pi i\left[\frac{1}{64} - \frac{1}{64}\right] = 0$$

Example 5.70

Using Cauchy's residue theorem, evaluate $\int_C \frac{2+3\sin\pi z}{z(z-1)^2}dz$, where C is a square bounded by the line $x = \pm 3$ and $y = \pm 3i$.

Solution:

Let $f(z) = \frac{2+3\sin\pi z}{z(z-1)^2}$

The poles of $f(z)$ are $z = 0$ and $z = 1$.

$z = 0$ is a simple pole.

and $z = 1$ is a pole of order 2.

Both these poles lie within C.

$\therefore$ Let $R_1 = [\text{Res} f(z)]_{z=0} = \lim_{z\to 0}(z-0), f(z)$

$$= \lim_{z\to 0}\left[\frac{2+3\sin\pi z}{(z-1)^2}\right] = 2$$

and $R_2 = [\text{Res} f(z)]_{z=1} = \frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}\left[(z-1)^2\cdot f(z)\right]$

$$= \lim_{z\to 1}\frac{d}{dz}\left[\frac{2+3\sin\pi z}{z}\right]$$

$$= \lim_{z\to 1}\left[\frac{(z)(3\pi\cos\pi z)-(2+3\sin\pi z)(1)}{z^2}\right]$$

$$= [(1)(-3\pi) - 2]$$

$$= -3\pi - 2.$$

$\therefore$ By Cauchy's residue theorem

$$\int_C f(z)dz = 2\pi i\,[R_1 + R_2]$$

$$= 2\pi i[2 - 3\pi - 2] = -6\pi^2 i$$

Exercise 5(c)

☞ Part - A

1. Define simple pole and multiple pole of a function of $f(z)$. Give one example of each.
2. Define removable singularity with an example.
3. Define essential singularity with an example.
4. Define isolated singularity with an example.
5. Define meromorphic function with an example.
6. Define entire function with an example.
7. State Cauchy's Integral formula.
8. State Cauchy's residue theorem.
9. State the formulas for finding the residue of a function at a simple pole.
10. State the formula for finding the residue of a function at a pole of order m.
11. Find the singularities of the following functions and classify the singularities.

    (a) $f(z) = \dfrac{e^{2z}}{(z-1)^3}$ (b) $f(z) = \dfrac{z}{(z+1)(z+2)}$

    (c) $f(z) = (z-3)\cdot\sin\dfrac{1}{z+2}$ (d) $f(z) = \dfrac{z-\sin z}{z^3}$

    (e) $f(z) = \sin\left(\dfrac{1}{1-z}\right)$ (f) $f(z) = \dfrac{1}{z(e^z-1)}$

    (g) $f(z) = (z-i)\sin\dfrac{1}{z+2i}$ (h) $f(z) = e^{-1/z^2}$

12. What kind of singularities do the following function have

    (a) $f(z)=\dfrac{1}{1-e^z}$ at $z=2\pi i$ (b) $f(z)=\dfrac{1}{\sin z-\cos z}$ at $z=\pi/4$

    (c) $f(z) = \tan 1/z$ at $z = 0$ (d) $f(z) = z\text{cosec}z$ at $z = \infty$

    (e) $f(z) = \dfrac{\sin(z-a)}{z-a}$ at $z = a$ (f) $f(z) = e^{1/z}$ at $z = 0$

13. Find the residue of $f(z)$ at $z = 0$, for the following function.

    (a) $f(z) = \dfrac{e^z}{z}$ (b) $f(z) = \dfrac{1-\sin z}{z^2}$

    (c) $f(z) = \dfrac{1-e^z}{z^3}$ (d) $f(z) = \dfrac{1}{z(z-2)(z+3)}$

    (e) $f(z) = \dfrac{z+1}{z-z^3}$ (f) $f(z) = \dfrac{z^2+2z+5}{z(z^2+1)}$

14. Calculate the residue of $f(z)$ for each of the following at its isolated singularities.

(a) $\dfrac{z^2}{z^2+0^2}$ (b) $\dfrac{z\sin z}{(z-\pi)^3}$ (c) $\dfrac{ze^z}{(z-1)^2}$ (d) $\dfrac{\cos z}{z^3}$

(e) $\dfrac{\cos z}{z}$ (f) $\dfrac{z^3}{(z+a)^2}$ (g) $\dfrac{z}{z^2+2z+5}$

15. Evaluate the following using Cauchy's residue theorem.

(a) $\displaystyle\int_C \frac{e^{-z}}{z^2}dz$ where C is $|z|=1$

(b) $\displaystyle\int_C \frac{3z^2+z+1}{(z-1)(z+3)}dz$ where C is $|z|=2$

(c) $\displaystyle\int_C \frac{dz}{z^3(z+4)}$ where C is $|z|=3$

(d) $\displaystyle\int_C \frac{dz}{\sin z}$ where C is $|z|=1$.

(e) $\displaystyle\int_C \frac{z^2}{z^2+1}dz$ where C is $|z|=2$

☞ **Part - B**

16. Find the Laurent's series of $f(z) = \dfrac{4-3z}{z(1-z)(2-z)}$ about $z=0$ and hence obtain the residue of $f(z)$ at $z=0$.

17. Evaluate $\displaystyle\int_C \frac{z^2+1}{z^2-1}dz$ if C is (i) $|z-1|=1$ (ii) $|z+1|=1$. Using Cauchy's residue theorem.

18. Evaluate $\displaystyle\int_C \frac{12z-7}{(z-1)^2(2z+3)}dz$ where C is $x^2+y^2=4$ using Cauchy's residue theorem.

19. Using Cauchy's residue theorem, evaluate the following:

(a) $\displaystyle\int_C \frac{\sin\pi z^2 + f\cos\pi z^2}{(z-1)(z-2)}dz$ where C is $|z|=3$.

(b) $\int_C \frac{3\cos z}{2i - 3z}$ where C is $|z| = 1$.

(c) $\int_C \frac{e^z dz}{\cosh z}$ where C is $|z| = 5$

(d) $\int_C ze^{1/z} dz$ where C is $|z| = 5$

(e) $\int_C \frac{e^{2z}}{z(z^2 + 2z + 2)} dz$ where C is $|z| = 3$

(f) $\int_C \frac{dz}{z^2(z+1)}$ where C is $|z| = \frac{1}{2}$

(g) $\int_C \frac{e^{2z}}{(z+1)^2} dz$ where C is $|z| = 2$

(h) $\int_C \frac{\sin^6 z}{(z - \pi/6)^2} dz$ where C is $|z| = 2$

(i) $\int_C \frac{e^z}{(z^2 + \pi^2)^2} dz$ where C is $|z| = 4$

20. Evaluate the following using Cauchy's residue theorem.

(a) $\int_C \frac{z+7}{z^2 + 2z + 5} dz$ where C is $|z - i| = 3/2$

(b) $\int_C \frac{dz}{z^3(z+4)}$ where C is $|z + 2| = 3$

(c) $\int_C \frac{e^z}{z^2 + \pi^2} dz$ where C is $|z - i| = 3$

(d) $\int_C \frac{z^2}{i(z-1)(z+1)^2} dz$ where C is $|z| = \sqrt{2}$.

(e) $\int_C \frac{3z^3 + 2}{(z-1)(z^2+4)} dz$ where C is $|z - 2| = 2$

(f) $\int_C \frac{2z^2 + z}{z^2 - 1} dz$ where C is $|z - 1| = 1$